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BASICS OF RIGHT NABLA FRACTIONAL CALCULUS ON TIME SCALES

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Abstract. We develop the right nabla fractional calculus on time scales. We introduce the related Riemann-Liouville type fractional integral and Caputo like fractional derivative and prove a fractional Taylor formula with integral remainder.

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1 Background

For the basics of times scales the reader is referred to [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

Let $\mathbb{T}$ be a time scale, and $\widehat{h}_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, such that $\forall\, s, t \in \mathbb{T}$, $\widehat{h}_0(t,s) = 1$,

$$\widehat{h}_{k+1}(t,s) = \int_s^t \widehat{h}_k(\tau,s)\,\nabla\tau. \tag{1.1}$$

Here $\widehat{h}_k$ are ld-continuous in t, and

$$\widehat{h}_k^{\nabla}(t,s) = \widehat{h}_{k-1}(t,s),\ k \in \mathbb{N}, t \in \mathbb{T}_k,$$

with $\widehat{h}_1(t,s) = t - s$, $\forall\, s, t \in \mathbb{T}$.

From [3], we write down Taylor's formula in terms of nabla polynomials:

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Theorem 1 *Assume that* $\mathbb{T} = \mathbb{T}_k$. *Let* $f \in C_{ld}^m(\mathbb{T}, \mathbb{R})$, $m \in \mathbb{N}$, $b, t \in \mathbb{T}$. *Then*

$$f(t) = \sum_{k=0}^{m-1} \widehat{h}_k(t,b) f^{\nabla^k}(b) + \int_b^t \widehat{h}_{m-1}(t, \rho(\tau)) f^{\nabla^m}(\tau) \nabla\tau. \tag{1.2}$$

Call

$$R_m^b(f)(t) := \int_b^t \widehat{h}_{m-1}(t, \rho(\tau)) f^{\nabla^m}(\tau) \nabla\tau = -\int_t^b \widehat{h}_{m-1}(t, \rho(\tau)) f^{\nabla^m}(\tau) \nabla\tau. \tag{1.3}$$

Following [3], we define

$$\widehat{g}_0(t,s) = 1,$$
$$\widehat{g}_{n+1}(t,s) = \int_s^t \widehat{g}_n(\rho(\tau), s) \nabla\tau, \;\; n \in \mathbb{N}, s, t \in \mathbb{T}. \tag{1.4}$$

Notice here

$$\widehat{g}_{n+1}^{\nabla}(t,s) = \widehat{g}_n(\rho(t), s), \;\; t \in \mathbb{T}_k,$$
$$\widehat{g}_1(t,s) = t - s, \;\; \forall\, s, t \in \mathbb{T}.$$

From [3] we need

Theorem 2 *If* $\mathbb{T} = \mathbb{T}_k = \mathbb{T}^k$, *and* $n \in \mathbb{N}_0$, *then*

$$\widehat{h}_n(t,s) = (-1)^n \widehat{g}_n(s,t), \;\; \forall\, s, t \in \mathbb{T}. \tag{1.5}$$

By Theorem 2 we get

$$R_m^b(f)(t) = (-1)^m \int_t^b \widehat{g}_{m-1}(\rho(\tau), t) f^{\nabla^m}(\tau) \nabla\tau. \tag{1.6}$$

We make

Definition 3 *Let* $\alpha \geq 0$ *real number. We consider the continuous functions*

$$\widehat{g}_\alpha : \mathbb{T}^2 \to \mathbb{R},$$

such that

$$\widehat{g}_0(t,s) = 1,$$
$$\widehat{g}_{\alpha+1}(t,s) = \int_s^t \widehat{g}_\alpha(\rho(\tau), s) \nabla\tau, \;\; \forall\, s, t \in \mathbb{T}. \tag{1.7}$$

We are motivated by the formula

$$\int_t^x \frac{(x-s)^{\mu-1}}{\Gamma(\mu)} \frac{(s-t)^{\nu-1}}{\Gamma(\nu)} ds = \frac{(x-t)^{\mu+\nu-1}}{\Gamma(\mu+\nu)}, \tag{1.8}$$

where $\mu, \nu > 0$ and Γ the gamma function.

We make

Assumption 4 *Let $\alpha, \beta > 1$ and $x < t \leq \tau$, $x, t, \tau \in \mathbb{T}$. We assume that*

$$\int_{x}^{\rho(\tau)} \widehat{g}_{\alpha-1}\left(\rho\left(t\right), x\right) \widehat{g}_{\beta-1}\left(\rho\left(\tau\right), t\right) \nabla t = \widehat{g}_{\alpha+\beta-1}\left(\rho\left(\tau\right), x\right). \tag{1.9}$$

We call for $\alpha, \beta > 1$ and $x < t \leq \tau$,

$$\gamma\left(x, \tau\right) := \int_{\rho(\tau)}^{\tau} \widehat{g}_{\alpha-1}\left(\rho\left(t\right), x\right) \widehat{g}_{\beta-1}\left(\rho\left(\tau\right), t\right) \nabla t.$$

It holds

$$\gamma\left(x, \tau\right) = \nu\left(\tau\right) \widehat{g}_{\alpha-1}\left(\rho\left(\tau\right), x\right) \widehat{g}_{\beta-1}\left(\rho\left(\tau\right), \tau\right), \tag{1.10}$$

where $\nu\left(\tau\right) := \tau - \rho\left(\tau\right)$, the backward graininess, see [9], p. 332, under the assumption $\mathbb{T} = \mathbb{T}_k$.

2 Results

We need

Definition 5 *Let $a, b \in \mathbb{T}$, $\alpha \geq 1$ and $f : [a, b] \cap \mathbb{T} \to \mathbb{R}$. Here $f \in L_1\left((a, b] \cap \mathbb{T}\right)$ (Lebesgue ∇-integrable function on $(a, b] \cap \mathbb{T}$). We define the right ∇-Riemann-Liouville type fractional integral*

$$J_{b-}^{\alpha} f\left(t\right) := \int_{t}^{b} \widehat{g}_{\alpha-1}\left(\rho\left(\tau\right), t\right) f\left(\tau\right) \nabla \tau, \tag{2.1}$$

for $t \in [a, b] \cap \mathbb{T}$. Here $\int_t^b \cdot \nabla \tau = \int_{(t,b]} \cdot \nabla \tau$.

By [8] we get that $J_{b-}^{1} f\left(t\right) = \int_t^b f\left(\tau\right) \nabla \tau$ is absolutely continuous in $t \in [a, b] \cap \mathbb{T}$.

Lemma 6 *Let $\alpha > 1$, $f \in L_1\left((a, b] \cap \mathbb{T}\right)$, $f : [a, b] \cap \mathbb{T} \to \mathbb{R}$. Assume that $\widehat{g}_{\alpha-1}\left(\rho\left(\tau\right), t\right)$ is Lebesgue ∇-measurable on $\left([a, b] \cap \mathbb{T}\right)^2$; $a, b \in \mathbb{T}$. Then $J_{b-}^{\alpha} f \in L_1\left([a, b] \cap \mathbb{T}\right)$, that is $J_{b-}^{\alpha} f$ is finite a.e.*

Proof. By Tietze's extension theorem of General Topology we easily derive that the continuous function $\widehat{g}_{\alpha-1}$ on $\left([a, b] \cap \mathbb{T}\right)^2$ is bounded, since its continuous extension $G_{\alpha-1}$ on $[a, b]^2$ is bounded. Notice that $\left([a, b] \cap \mathbb{T}\right)^2$ is a closed subset of $[a, b]^2$.

So there exists $M > 0$ such that $\left|\widehat{g}_{a-1}\left(s, t\right)\right| \leq M, \forall \left(s, t\right) \in \left([a, b] \cap \mathbb{T}\right)^2$.

Let id denote the identity map. We see that

$\left(\rho, id\right)\left(\left((a, b] \cap \mathbb{T}\right) \times \left([a, b] \cap \mathbb{T}\right)\right) \subseteq \left([a, b] \cap \mathbb{T}\right)^2$.

Therefore $\left|\widehat{g}_{\alpha-1}\left(\rho\left(\tau\right), t\right)\right| \leq M$, $\forall \left(\tau, t\right) \in \left((a, b] \cap \mathbb{T}\right) \times \left([a, b] \cap \mathbb{T}\right)$, since $\left(\rho\left(\tau\right), t\right) \in \left([a, b] \cap \mathbb{T}\right)^2$.

Define $K : \Omega := \left([a, b] \cap \mathbb{T}\right)^2 \to \mathbb{R}$, by

$$K\left(\tau, t\right) := \begin{cases} \widehat{g}_{\alpha-1}\left(\rho\left(\tau\right), t\right), \text{ if } a \leq t < \tau \leq b, \\ 0, \text{ if } a \leq \tau \leq t \leq b, \end{cases}$$

where $t, \tau \in \mathbb{T}$.

Clearly K is Lebesgue ∇-measurable on Ω, since the restriction of a measurable function to a measurable subset of its domain is measurable function and the union of two measurable functions over disjoint domains is measurable. Notice that $|K(\tau,t)| \leq M, \forall (\tau,t) \in ([a,b] \cap \mathbb{T})^2$.

Next we consider the repeated double Lebesgue ∇-integral

$$\int_a^b \left(\int_a^b |K(\tau,t)| \, |f(\tau)| \nabla t \right) \nabla \tau = \int_a^b |f(\tau)| \left(\int_a^b |K(\tau,t)| \nabla t \right) \nabla \tau$$

$$\leq M(b-a) \int_a^b |f(\tau)| \nabla \tau = M(b-a) \|f\|_{L_1((a,b] \cap \mathbb{T})} < \infty.$$

By Tonelli's theorem we derive that $(\tau,t) \to K(\tau,t) f(\tau)$ is Lebesgue ∇-integrable over Ω.

Let now the characteristic function

$$\chi_{(t,b]}(\tau) = \begin{cases} 1, & \text{if } \tau \in (t,b] \\ 0, & \text{else,} \end{cases}$$

where $\tau \in [a,b] \cap \mathbb{T}$.

Then the function $(\tau,t) \to \chi_{(t,b]}(\tau) K(\tau,t) f(\tau)$ is Lebesgue ∇-integrable over Ω.

Hence by Fubini's theorem we get that

$$\int_a^b \chi_{(t,b]}(\tau) K(\tau,t) f(\tau) \nabla \tau = \int_t^b \widehat{g}_{\alpha-1}(\rho(\tau),t) f(\tau) \nabla \tau = J_{b-}^{\alpha} f(t)$$

is Lebesgue ∇-integrable on $[a,b] \cap \mathbb{T}$, proving the claim. □

We make

Assumption 7 *From now on we assume that $\widehat{g}_{\alpha-1}(\rho(\cdot),\cdot)$ is continuous on $([a,b] \cap \mathbb{T})^2$, for any $\alpha > 1$.*

We give

Definition 8 *Let $f \in L_1((a,b] \cap \mathbb{T})$. We define the right backward graininess deviation functional of f as follows*

$$\theta(f,\alpha,\beta,b,\mathbb{T},x) := \int_x^b f(\tau) \gamma(x,\tau) \nabla \tau. \tag{2.2}$$

It holds

$$\theta(f,\alpha,\beta,b,\mathbb{T},x) = \int_x^b f(\tau) \nu(\tau) \widehat{g}_{\alpha-1}(\rho(\tau),x) \widehat{g}_{\beta-1}(\rho(\tau),\tau) \nabla \tau, \tag{2.3}$$

under the assumption $\mathbb{T} = \mathbb{T}_k$.

If $\mathbb{T} = \mathbb{R}$, then $\theta(f,\alpha,\beta,b,\mathbb{T},x) = 0$.

We give the following semigroup property of right ∇-Riemann-Liouville type fractional integrals.

Theorem 9 *Let the time scale* $\mathbb{T}$ *such that* $a,b \in \mathbb{T}$, $f \in L_1((a,b] \cap \mathbb{T})$; $\alpha, \beta > 1$. *Then*

$$J_{b-}^{\alpha} J_{b-}^{\beta} f(x) = J_{b-}^{\alpha+\beta} f(x) + \theta(f, \alpha, \beta, b, \mathbb{T}, x), \ \ \forall\, x \in [a,b] \cap \mathbb{T}. \tag{2.4}$$

Proof. For $\beta > 1$ we have

$$J_{b-}^{\beta} f(t) = \int_t^b \widehat{g}_{\beta-1}(\rho(\tau), t) f(\tau) \nabla \tau.$$

We observe that

$$J_{b-}^{\alpha} J_{b-}^{\beta} f(x) = \int_x^b \widehat{g}_{\alpha-1}(\rho(t), x) J_{b-}^{\beta} f(t) \nabla t =$$

$$\int_x^b \widehat{g}_{\alpha-1}(\rho(t), x) \left(\int_t^b \widehat{g}_{\beta-1}(\rho(\tau), t) f(\tau) \nabla \tau \right) \nabla t =$$

$$\int_x^b \left(\int_t^b \widehat{g}_{\alpha-1}(\rho(t), x) \widehat{g}_{\beta-1}(\rho(\tau), t) f(\tau) \nabla \tau \right) \nabla t =: (*).$$

Clearly here it holds

$$|\widehat{g}_{\alpha-1}(\rho(t), x)| \leq M_1, \ \ \forall\, t, x \in [a,b] \cap \mathbb{T},$$

and

$$|\widehat{g}_{\beta-1}(\rho(\tau), t)| \leq M_2, \ \ \forall\, \tau, t \in [a,b] \cap \mathbb{T},$$

where $M_1, M_2 > 0$.

Hence

$$\left| J_{b-}^{\alpha} J_{b-}^{\beta} f(x) \right| \leq \int_x^b \left(\int_t^b |\widehat{g}_{\alpha-1}(\rho(t), x)| \, |\widehat{g}_{\beta-1}(\rho(\tau), t)| \, |f(\tau)| \nabla \tau \right) \nabla t \leq$$

$$M_1 M_2 \left(\int_x^b \left(\int_t^b |f(\tau)| \nabla \tau \right) \nabla t \right) \leq M_1 M_2 \left(\int_x^b \left(\int_a^b |f(\tau)| \nabla \tau \right) \nabla t \right) \leq$$

$$M_1 M_2 (b-a) \|f\|_{L_1((a,b] \cap \mathbb{T})} < \infty.$$

Therefore $J_{b-}^{\alpha} J_{b-}^{\beta} f(x)$ exists, $\forall\, x \in [a,b] \cap \mathbb{T}$. Consequently by Fubini's theorem we have

$$(*) = \int_x^b \left(\int_x^{\tau} \widehat{g}_{\alpha-1}(\rho(t), x) \widehat{g}_{\beta-1}(\rho(\tau), t) f(\tau) \nabla t \right) \nabla \tau =$$

$$\int_x^b f(\tau) \left(\int_x^{\tau} \widehat{g}_{\alpha-1}(\rho(t), x) \widehat{g}_{\beta-1}(\rho(\tau), t) \nabla t \right) \nabla \tau$$

$(x < t \leq \tau)$

$$\stackrel{(1.9)}{=} \int_x^b f(\tau) \left(\widehat{g}_{\alpha+\beta-1}(\rho(\tau), x) + \int_{\rho(\tau)}^{\tau} \widehat{g}_{\alpha-1}(\rho(t), x) \widehat{g}_{\beta-1}(\rho(\tau), t) \nabla t \right) \nabla \tau$$

$$= \int_x^b \widehat{g}_{\alpha+\beta-1}(\rho(\tau), x) f(\tau) \nabla \tau + \int_x^b f(\tau) \gamma(x, \tau) \nabla \tau$$

$$= J_{b-}^{\alpha+\beta} f(x) + \int_x^b f(\tau) \gamma(x, \tau) \nabla \tau.$$

So we have that

$$J_{b-}^{\alpha} J_{b-}^{\beta} f(x) = J_{b-}^{\alpha+\beta} f(x) + \int_x^b f(\tau)\,\gamma(x,\tau)\,\nabla\tau$$
$$= J_{b-}^{\alpha+\beta} f(x) + \theta(f,\alpha,\beta,b,\mathbb{T},x),$$

proving the claim. □

We make

Remark 10 *Let $\mu > 2 : m-1 < \mu \le m \in \mathbb{N}$, i.e. $m = \lceil \mu \rceil$ (ceiling of number), $\widetilde{\nu} = m - \mu$ ($0 \le \widetilde{\nu} < 1$). Let $f \in C_{ld}^{m}([a,b] \cap \mathbb{T})$. Clearly here ([10]) f^{∇^m} is a Lebesgue ∇-integrable function. We define the right nabla fractional derivative on $\mathbb{T}$ of order $\mu - 1$ as follows:*

$$\nabla_{b-}^{\mu-1} f(t) = (-1)^m \left(J_{b-}^{\widetilde{\nu}+1} f^{\nabla^m} \right)(t) = (-1)^m \int_t^b \widehat{g}_{\widetilde{\nu}}(\rho(\tau),t)\, f^{\nabla^m}(\tau)\,\nabla\tau, \tag{2.5}$$

$\forall\, t \in [a,b] \cap \mathbb{T}$.

Notice $\nabla_{b-}^{\mu-1} f \in C([a,b] \cap \mathbb{T})$, by a simple argument using the dominated convergence theorem in Lebesgue ∇-sense.

If $\mu = m$, then $\widetilde{\nu} = 0$, then

$$\nabla_{b-}^{m-1} f(t) = (-1)^m \int_t^b f^{\nabla^m}(\tau)\,\nabla\tau = (-1)^m \left(f^{\nabla^{m-1}}(b) - f^{\nabla^{m-1}}(t) \right). \tag{2.6}$$

More generally, by [8], given that $f^{\nabla^{m-1}}$ is everywhere finite and absolutely continuous on $[a,b] \cap \mathbb{T}$, then f^{∇^m} exists ∇-a.e. and is Lebesgue ∇-integrable on $(t,b] \cap \mathbb{T}$, $\forall\, t \in [a,b] \cap \mathbb{T}$, and one can plug it into (2.5).

Remark 11 *We observe that*

$$J_{b-}^{\mu-1} \nabla_{b-}^{\mu-1} f(t) = (-1)^m \left(J_{b-}^{\mu-1} J_{b-}^{\widetilde{\nu}+1} f^{\nabla^m}(t) \right)$$
$$\overset{(2.4)}{=} (-1)^m \left(J_{b-}^{\mu-1+\widetilde{\nu}+1} f^{\nabla^m}(t) + \theta\left(f^{\nabla^m}, \mu-1, \widetilde{\nu}+1, b, \mathbb{T}, t\right) \right)$$
$$= (-1)^m \left(J_{b-}^{m} f^{\nabla^m}(t) + \theta\left(f^{\nabla^m}, \mu-1, \widetilde{\nu}+1, b, \mathbb{T}, t\right) \right). \tag{2.7}$$

Hence we proved that

$$J_{b-}^{\mu-1} \nabla_{b-}^{\mu-1} f(t) + (-1)^{m+1}\, \theta\left(f^{\nabla^m}, \mu-1, \widetilde{\nu}+1, b, \mathbb{T}, t\right) =$$
$$(-1)^m \left(J_{b-}^{m} f^{\nabla^m}(t) \right) = (-1)^m \left(\int_t^b \widehat{g}_{m-1}(\rho(\tau),t)\, f^{\nabla^m}(\tau)\,\nabla\tau \right)$$
$$\overset{(1.6)}{=} \left(R_m^b(f) \right)(t), \tag{2.8}$$

under the assumption $\mathbb{T} = \mathbb{T}_k = \mathbb{T}^k$.

We have established the following right nabla time scales Taylor formula.

Theorem 12 *Assume* $\mathbb{T} = \mathbb{T}_k = \mathbb{T}^k$. *Let* $f \in C_{ld}^m(\mathbb{T})$, $m \in \mathbb{N}$, $a, b \in \mathbb{T}$, *with* $\mu > 2 : m - 1 < \mu \leq m$, $\widetilde{\nu} = m - \mu$. *Then*

$$f(t) = \sum_{k=0}^{m-1} \widehat{h}_k(t, b) f^{\nabla^k}(b) + J_{b-}^{\mu-1} \nabla_{b-}^{\mu-1} f(t) +$$

$$(-1)^{m+1} \theta\left(f^{\nabla^m}, \mu - 1, \widetilde{\nu} + 1, b, \mathbb{T}, t\right), \tag{2.9}$$

$\forall\, t \in [a, b] \cap \mathbb{T}$.

Remark 13 *One can rewrite (2.9) as follows*

$$f(t) = \sum_{k=0}^{m-1} \widehat{h}_k(t, b) f^{\nabla^k}(b) +$$

$$(-1)^{m+1} \int_t^b f^{\nabla^m}(\tau)\, \nu(\tau)\, \widehat{g}_{\mu-2}(\rho(\tau), t)\, \widehat{g}_{\widetilde{\nu}}(\rho(\tau), \tau)\, \nabla\tau$$

$$+ \int_t^b \widehat{g}_{\mu-2}(\rho(\tau), t) \left(\nabla_{b-}^{\mu-1} f\right)(\tau)\, \nabla\tau, \tag{2.10}$$

$\forall\, t \in [a, b] \cap \mathbb{T}$.

Corollary 14 *In the assumptions of Theorem 12, additionally assume that* $f^{\nabla^k}(b) = 0$, $k = 0, 1, ..., m - 1$. *Then*

$$A(t) := f(t) + (-1)^m \theta\left(f^{\nabla^m}, \mu - 1, \widetilde{\nu} + 1, b, \mathbb{T}, t\right)$$

$$= \int_t^b \widehat{g}_{\mu-2}(\rho(\tau), t) \left(\nabla_{b-}^{\mu-1} f\right)(\tau)\, \nabla\tau, \tag{2.11}$$

$\forall\, t \in [a, b] \cap \mathbb{T}$.

Remark 15 *Notice (by [8]) that* $\left(J_{b-}^{\mu-1} \nabla_{b-}^{\mu-1} f\right)(t)$ *and* $\theta(f^{\nabla^m}, \mu - 1, \widetilde{\nu} + 1, b, \mathbb{T}, t)$ *are absolutely continuous functions on* $[a, b] \cap \mathbb{T}$.

One can use (2.10) and (2.11) to establish right fractional nabla inequalities on time scales of Poincaré type, Sobolev type, Opial type, Ostrowski type and Hilbert-Pachpatte type, etc, analogous to [1]. To keep the article short we avoid this similar task.

Our theory is not void because it is valid when $\mathbb{T} = \mathbb{R}$, see also [1].

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