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EXISTENCE OF SOLUTIONS TO A NON-AUTONOMOUS ABSTRACT NEUTRAL DIFFERENTIAL EQUATION WITH DEVIATED ARGUMENT

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Abstract. We study the existence, uniqueness and asymptotic stability of solutions to a neutral differential equation with a deviated argument in an arbitrary Banach space. The results are obtained by applying the Sobolevskiĭ-Tanabe theory of parabolic equations, fractional powers of operators and the Banach fixed point theorem. As an application, we include an example to illustrate the theory.

Keywords: Analytic semigroup, Parabolic equation, Neutral differential equation with a deviated argument, Banach fixed point theorem.

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1 Introduction

The purpose of this paper is to study the following neutral differential equation with a deviated argument considered in a Banach space X:

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}[u(t)+g(t,u(a(t)))] + A(t)\,[u(t)+g(t,u(a(t)))] & \\ = \ & f(t,u(t),u([h(u(t),t)])),\ t>0; \\ u(0) \ = \ & u_0. \end{aligned}\right\} \tag{1.1}$$

Here, we assume that $-A(t)$, for each $t \geq 0$, generates an analytic semigroup of bounded linear operators on X. The nonlinear functions f, g and h satisfy suitable growth conditions in their arguments stated in Section 2 and $a : [0,T] \to [0,T]$ satisfies the delay property.

The existence, uniqueness and continuous dependence of solutions to differential equations with deviated arguments have been studied by many authors (see e.g. Gal [4, 5]; Grimm [6]; Jankowski [10]; Muslim and Bahuguna [14]; Oberg [15]). But the complete theory is yet to be developed. We refer to Gal [4, 5], Grimm [6], Jankowski [10], Kwaspisz [12] and references cited therein for further details on differential equations with deviated arguments.

Adimy *et al* [1] have studied the existence and stability of a solution of the following general class of nonlinear partial neutral functional differential equations:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(u(t)-g(t,u_t)) &= A(u(t)-g(t,u_t)) + f(t,u_t),\ \ t \geq 0, \\ u_0 &= \varphi \in C_0, \end{aligned}$$

where the operator A is the Hille-Yosida operator not necessarily densely defined on the Banach space B and C_0 is an appropriate phase space (every element of C_0 is a function mapping $(-\infty, 0]$ into a Banach space B). The functions g and f are uniformly Lipschitz continuous in the second variable from $[0,\infty) \times C_0$ into B.

Muslim and Bahuguna [14] have proved the existence and uniqueness of a global solution to Problem (1.1) in a Banach for $A(t) = A$.

Our objective here is to establish the existence, uniqueness and asymptotic stability of solutions to Problem (1.1). We shall use the Banach fixed point theorem and the Sobolevskiĭ-Tanabe theory of parabolic equations to prove the existence and uniqueness of a solution to Problem (1.1).

The results presented here can be applied to Problem (1.1) with a nonlocal condition under some modified assumptions on the functions $f,\ g,\ h$ and the operator $A(t)$.

The paper is organized as follows. In Section 2, we will provide preliminaries, assumptions and lemmas that will be needed for proving our main results. We will prove the local and global existence of a solution in Section 3. In Section 4, we will discuss the asymptotic stability of a solution. Finally, we will provide an example to illustrate the application of the abstract results.

2 Preliminaries and Assumptions

In this section, we will introduce assumptions, preliminaries and lemmas that will be used to prove our main results. We briefly outline the facts concerning analytic semigroups, fractional powers of operators, and the homogeneous and inhomogeneous linear Cauchy initial value problems. The material presented here is covered in more detail by Friedman [2], Henry [7], Krien [11], Ladas and Lakshmikantham [13], Sobolevskiĭ [16] and Tanabe [17].

Let X be a complex Banach space with norm $\|\cdot\|$. Let $T \in [0,\infty)$ and $\{A(t) : 0 \le t \le T\}$ be a family of closed linear operators on the Banach space X. We shall make use of the following assumptions.

(A1) The domain $D(A)$ of $A(t)$ is dense in X and independent of t.

(A2) For each $t \in [0,T]$, the resolvent $R(\lambda; A(t))$ exists for all Re $\lambda \le 0$ and there is a constant $C > 0$ (independent of t and λ) such that

$$\|R(\lambda; A(t))\| \le \frac{C}{|\lambda|+1}, \text{ Re } \lambda \le 0, \ t \in [0,T].$$

(A3) For each fixed $s \in [0,T]$, there are constants $C > 0$ and $\rho \in (0,1]$, such that

$$\|[A(t) - A(\tau)]\, A(s)^{-1}\| \le C\, |t-\tau|^{\rho},$$

for any $t, \tau \in [0,T]$. Here C and ρ are independent of t, τ and s.

Assumption **(A2)** implies that for each $s \in [0,T]$, $-A(s)$ generates a strongly continuous analytic semigroup $\{e^{-t\,A(s)} : t \ge 0\}$ in $\mathcal{L}(X)$, where $\mathcal{L}(X)$ denotes the Banach algebra of all bounded linear operators on X [2, 16]. Then there exist positive constants C and d such that

$$\|e^{-t\,A(s)}\| \le Ce^{-d\,t}, \quad t \ge 0; \tag{2.1}$$

$$\|A(s)\, e^{-tA(s)}\| \le \frac{Ce^{-d\,t}}{t}, \quad t > 0, \tag{2.2}$$

for all $s \in [0,T]$. It is to be noted that assumption **(A3)** implies that there exists a constant $C > 0$ such that

$$\|A(t)\, A(s)^{-1}\| \le C, \tag{2.3}$$

for all $0 \le s, t \le T$. Hence, for each t, the functional $y \mapsto \|A(t)y\|$ defines an equivalent norm on $D(A) = D(A(0)) \equiv X_1$ and the mapping $t \mapsto A(t)$ from $[0,T]$ into $\mathcal{L}(X_1, X)$ is uniformly Hölder continuous.

Consider the following homogeneous Cauchy problem,

$$\frac{\mathrm{d}u}{\mathrm{d}t} + A(t)\, u = 0; \quad u(t_0) = u_0. \tag{2.4}$$

Then the solution to the homogeneous Cauchy problem (2.4) is given by the following Theorem.

Theorem 2.1 *[2, 16] Let the assumptions **(A1)**, **(A2)** and **(A3)** hold. Then there exists a unique fundamental solution* $\{U(t,s) : 0 \le s \le t \le T\}$ *to* (2.4) *that possesses the following properties:*

(i) $U(t,s) \in \mathcal{L}(X)$ *and* $U(t,s)$ *is strongly continuous in* t, s *for all* $0 \le s \le t \le T$.

(ii) $U(t,s)\, x \in D(A)$ *for each* $x \in X$, *for all* $0 \le s \le t \le T$.

(iii) $U(t,r)\,U(r,s) = U(t,s)$ *for all* $0 \le s \le r \le t \le T$.

(iv) *The derivative* $\partial U(t,s)/\partial t$ *exists in the strong operator topology and belongs to* $\mathcal{L}(X)$ *for all* $0 \le s < t \le T$, *and is strongly continuous in* t *and* s, *where* $0 \le s < t \le T$.

(v) $\dfrac{\partial U(t,s)}{\partial t} + A(t)\,U(t,s) = 0$ *and* $U(s,s) = I$ *for all* $0 \le s < t \le T$.

For $t_0 \ge 0$, let $C^\beta([t_0,T];X)$ denote the space of all X-valued functions $h(t)$, that are uniformly Hölder continuous on $[t_0,T]$ with exponent β, where $0 < \beta \le 1$. Define

$$[h]_\beta = \sup_{t,s\in[t_0,T],\ t\neq s} \frac{\|h(t)-h(s)\|}{|t-s|^\beta}.$$

Then $C^\beta([t_0,T];X)$ is a Banach space with respect to the norm

$$\|h\|_{C^\beta([t_0,T];X)} = \sup_{t_0\le t\le T} \|h(t)\| + [h]_\beta.$$

Consider the following inhomogeneous Cauchy problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} + A(t)\,u = f(t), \quad u(t_0) = u_0. \tag{2.5}$$

Theorem 2.1 *[2, 16] Let the assumptions* ***(A1)-(A3)*** *hold. If* $f \in C^\beta([t_0,T];X)$, *then there exists a unique solution of (2.5). Furthermore, the solution can be written as*

$$u(t) = U(t,t_0)\,u_0 + \int_{t_0}^{t} U(t,s)\,f(s)\,\mathrm{d}s, \quad t_0 \le t \le T,$$

and $u : [t_0,T] \to X$ *is strongly continuously differentiable on* $(t_0,T]$.

The bound in (2.1) allows us to define negative fractional powers of the operator $A(t)$. For $\alpha > 0$, define negative fractional powers $A(t)^{-\alpha}$ by the formula,

$$A(t)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-\tau A(t)}\,\tau^{\alpha-1}\,\mathrm{d}\tau.$$

Then $A(t)^{-\alpha}$ is one-to-one and bounded linear operator on X. Thus, there exists an inverse of the operator $A(t)^{-\alpha}$. We define the positive fractional powers of $A(t)$ by $A(t)^\alpha \equiv [A(t)^{-\alpha}]^{-1}$. Then $A(t)^\alpha$ is closed linear operator with dense domain $D(A(t)^\alpha)$ in X and $D(A(t)^\alpha) \subset D(A(t)^\beta)$ if $\alpha > \beta$. For $0 < \alpha \le 1$, let $X_\alpha = D(A(0)^\alpha)$ and equip this space with the graph norm

$$\|x\|_\alpha = \|A(0)^\alpha\,x\|.$$

Then $(X_\alpha, \|\cdot\|_\alpha)$ is a Banach space. If $0 < \alpha \le 1$, the embedding $X_1 \hookrightarrow X_\alpha \hookrightarrow X$ are dense and continuous. For each $\alpha > 0$, define $X_{-\alpha} = (X_\alpha)^*$, the dual space of X_α, and endow it with the natural norm

$$\|x\|_{-\alpha} = \|A(0)^{-\alpha}x\|.$$

Let f, g and h be three continuous functions. For $0 < \alpha \le 1$, let W_α and $W_{\alpha-1}$ be open sets in X_α and $X_{\alpha-1}$, respectively. For each $u' \in W_\alpha$ and $u'' \in W_{\alpha-1}$, there are balls such that $B_\alpha(u',r') \subset W_\alpha$ and $B_{\alpha-1}(u'',r'') \subset W_{\alpha-1}$. We will make use of the following assumptions.

(A4) There exist constants $L_f = L_f(t, u', u'', r', r'') > 0$ and $0 < \theta_1 \leq 1$, such that the nonlinear map $f : [0,T] \times W_\alpha \times W_{\alpha-1} \to X$ satisfies

$$\|f(t,x,x') - f(s,y,y')\| \leq L_f(|t-s|^{\theta_1} + \|x-y\|_\alpha + \|x'-y'\|_{\alpha-1}), \tag{2.6}$$

for all $x, y \in B_\alpha$, $x', y' \in B_{\alpha-1}$ and for all $s, t \in [0,T]$.

(A5) There exist constants $L_h = L_h(u', t, r') > 0$ and $0 < \theta_2 \leq 1$, such that $h(\cdot, 0) = 0$, $h : W_\alpha \times [0,T] \to [0,T]$ satisfies

$$|h(x,t) - h(y,s)| \leq L_h\,(\|x-y\|_\alpha + |t-s|^{\theta_2}), \tag{2.7}$$

for all $x, y \in B_\alpha$ and for all $s, t \in [0,T]$.

(A6) Let $0 \leq \alpha < \beta < 1$. There exists constant $L_g = L_g(t, u'', r'', \beta) > 0$ such that the continuous function $g : [0,T] \times W_{\alpha-1} \to X_\beta$ satisfies

$$\begin{aligned} \|g(t,x) - g(s,y)\|_\beta &\leq L_g\{|t-s| + \|x-y\|_{\alpha-1}\}, \\ 4\,L_g\,\|A(0)^{\alpha-\beta-1}\| &< 1 \end{aligned} \tag{2.8}$$

for all $x, y \in B_{\alpha-1}$ and $t, s \in [0,T]$.

(A7) The function $a : [0,T] \to [0,T]$ satisfies the following two conditions:

(i) a satisfies the delay property $a(t) \leq t$ for all $t \in [0,T]$;

(ii) The function a is Lipschitz continuous; that is, there exists a positive constant L_a such that

$$|a(t) - a(s)| \leq L_a\,|t-s|,\ \text{for all } t, s \in [0,T] \quad \text{and} \quad 1 > \|A(0)^{-1}\|\,L_a.$$

The following lemmas will be used in the subsequent sections.

Lemma 2.2 *[3, Lemma 1.1] Let $h \in C^\beta([t_0,T];X)$. Define $H : C^\beta([t_0,T];X) \to C([t_0,T];X_1)$ by*

$$H\,h(t) = \int_{t_0}^{t} U(t,s)\,h(s)\,\mathrm{d}s,\ t_0 \leq t \leq T.$$

Then H is a bounded mapping and $\|H\,h\|_{C([t_0,T];X_1)} \leq C\|h\|_{C^\beta([t_0,T];X)}$, for some $C > 0$.

We have the following corollary from Lemma 2.2.

Corollary 2.2 *For $y \in X_1$, define*

$$P(y;h) = U(t,0)y + \int_0^t U(t,s)\,h(s)\,\mathrm{d}s,\ 0 \leq t \leq T.$$

Then P is a bounded linear mapping from $X_1 \times C^\beta([t_0,T];X)$ into $C([t_0,T];X_1)$.

Lemma 2.3 *[8, Lemma 2] Let $0 < \alpha \leq 1$ and $f \in C([t_0,T];X_\alpha)$. Define*

$$w(t) = \int_{t_0}^{t} U(t,s)\,f(s)\,\mathrm{d}s,\ t_0 \leq t \leq T.$$

Then $w \in C([t_0,T];X_1) \cap C^1((t_0,T];X)$ and $w'(t) + A(t)\,w(t) = f(t),\ t_0 < t \leq T$.

We will make use of the following Lemma [18, Lemma 2.2] to prove the existence of a global solution to (1.1).

Lemma 2.4 *Let w be a continuous function from $[0, t_1]$ into $\mathbb{R}^+$. Let $a, b, c \geq 0$, $\omega, \rho \in \mathbb{R}$, $0 < \alpha < 1$ and let*

$$w(t) \leq a\, e^{\omega t} + b \int_0^t e^{\rho(t-s)}\, (t-s)^{-\alpha}\, w(s)\, \mathrm{d}s + c \int_0^t e^{\omega(t-s)}\, w(s)\, \mathrm{d}s,$$

for $0 \leq t \leq t_1$. Then, for every real γ such that $\gamma > \max\{\omega, \rho\}$ and $k = b\,\Gamma(1-\alpha)\,(\gamma-\rho)^{\alpha-1} + c\,(\gamma-\omega)^{-1} < 1$, we have

$$w(t) \leq a\,(1-k)^{-1} e^{\gamma t},\ 0 \leq t \leq t_1.$$

3 Existence of Solution

In this section, we will establish the existence and uniqueness of a local solution to (1.1). Let $I = [0, \delta]$ for some positive number δ to be specified later. Let $\mathcal{C}_\alpha$, $0 \leq \alpha \leq 1$ denote the space of all X_α-valued continuous functions on I, endowed with the sup-norm, $\sup\limits_{t \in I} \|\psi(t)\|_\alpha$, $\psi \in C(I; X_\alpha)$. Let

$$Y_\alpha \equiv C_{L_\alpha}(I; X_{\alpha-1}) = \{\psi \in \mathcal{C}_\alpha : \|\psi(t) - \psi(s)\|_{\alpha-1} \leq L_\alpha\, |t-s|,\ \text{for all } t, s \in I\},$$

where L_α is a positive constant to be specified later. It is clear that Y_α is a Banach space endowed with the sup-norm of $\mathcal{C}_\alpha$.

Definition 3.1 *A continuous function $u \in C_{L_\alpha}(I; X_{\alpha-1})$ is said to be a mild solution to Problem (1.1) if u is the solution of the following integral equation*

$$\begin{aligned} u(t) &= U(t,0)\,[u(0) + g(0, u_0)] - g(t, u(a(t))) \\ &\quad + \int_0^t U(t,s)\, f(s, u(s), u([h(u(s), s)]))\, \mathrm{d}s,\ t \in I \end{aligned} \tag{3.1}$$

and satisfies the initial condition $u(0) = u_0$.

Definition 3.2 *Given $u_0 \in X_\alpha$, by a solution of Problem (1.1), we mean a function $u : I \to X$ that satisfies:*

(i) $u(\cdot) + g(\cdot, u(a(\cdot))) \in C_{L_\alpha}(I; X_\alpha) \cap C^1((0,\delta); X) \cap C(I; X)$;

(ii) $u(t) \in X_1$, *for all* $t \in (0, \delta)$;

(iii) $\dfrac{\mathrm{d}}{\mathrm{d}t}[u(t) + g(t, u(a(t)))] + A(t)\,[u(t) + g(t, u(a(t)))] = f(t, u(t), u([h(u(t), t)]))$, *for all* $t \in (0, \delta)$;

(iv) $u(0) = u_0$.

Let $u_0 \in X_\alpha$ and let $r > 0$ be chosen small enough so that the assumptions of **(A4)–(A6)** hold for the closed balls $B_\alpha = B_\alpha(u_0, r)$ and $B_{\alpha-1} = B_{\alpha-1}(u_0, r)$. Let $K > 0$ and $0 < \eta < \beta - \alpha$ be fixed constants. Let

$$\begin{aligned}\mathcal{S}_\alpha = \big\{ & y \in \mathcal{C}_\alpha \cap Y_\alpha : y(0) = u_0, \\ & \sup_{t \in I} \|y(t) - u_0\|_\alpha \le r, \|y(t) - y(s)\|_\alpha \le K|t - s|^\eta \text{ for all } s, t \in I \big\}.\end{aligned}$$

Then $\mathcal{S}_\alpha$ is a non-empty, closed and bounded subset of $\mathcal{C}_\alpha$.

Now we will prove the following theorem concerning the existence and uniqueness of a local mild solution to (1.1). The proof is based on ideas of Friedman [2] and Gal [4].

Theorem 3.3 *Let $u_0 \in X_\beta$, where $0 < \alpha < \beta \le 1$ and the assumptions (A1)-(A7) hold. Then there exists a positive number $\delta = \delta(\alpha, u_0)$ and a unique local solution $u(t)$ of Problem (1.1) on the interval $[0, \delta]$.*

Proof. Let $v \in \mathcal{S}_\alpha$. Put $f_v(t) = f(t, v(t), v([h(v(t), t)]))$. Then the assumptions **(A4)** and **(A5)** imply that f_v is Hölder continuous on I of exponent $\gamma = \min\{\theta_1, \theta_2, \eta\}$. Define a map F by

$$Fv(t) = U(t, 0)\,[u_0 + g(0, u_0)] - g(t, v(a(t))) + \int_0^t U(t, s)\, f_v(s)\, \mathrm{d}s, \quad t \in [0, \delta]. \tag{3.2}$$

From Lemma 2.2, it is clear that the map F is well defined and $Fv \in \mathcal{C}_\alpha$. We will claim that F maps from $\mathcal{S}_\alpha$ into itself, for sufficiently small $\delta > 0$. Indeed, if $t_1, t_2 \in I$ with $t_2 > t_1$, then we have

$$\begin{aligned}\|Fv(t_2) - Fv(t_1)\|_{\alpha-1} \le {} & \|[U(t_2, 0) - U(t_1, 0)]\,[u_0 + g(0, u_0)]\|_{\alpha-1} \\ & + \|g(t_2, v(a(t_2))) - g(t_1, v(a(t_1)))\|_{\alpha-1} \\ & + \left\| \int_0^{t_2} U(t_2, s)\, f_v(s)\, \mathrm{d}s - \int_0^{t_1} U(t_1, s)\, f_v(s)\, \mathrm{d}s \right\|_{\alpha-1}.\end{aligned} \tag{3.3}$$

We will use the bounded inclusion $X \subset X_{\alpha-1}$ to estimate the first and third term on the right hand side of (3.3). The first term on the right hand side of (3.3) is estimated as follows [2, see Lemma II.14.1],

$$\|[U(t_2, 0) - U(t_1, 0)]\,[u_0 + g(0, u_0)]\|_{\alpha-1} \le C_1(u_0, g(0, u_0))\,(t_2 - t_1), \tag{3.4}$$

where C_1 is some positive constant.

Making use of assumptions **(A6)** and **(A7)**, we get

$$\|g(t_2, v(a(t_2))) - g(t_1, v(a(t_1)))\|_{\alpha-1} \le C_2\,|t_2 - t_1| \tag{3.5}$$

where $C_2 = \left\|A(0)^{\alpha-\beta-1}\right\|\, L_g\,(1 + L_a\, L_\alpha)$.

For the last term on the right hand side of (3.3), we use [2, Lemma II. 14.4] to obtain the following estimate,

$$\left\| \int_0^{t_2} U(t_2, s)\, f_v(s)\, \mathrm{d}s - \int_0^{t_1} U(t_1, s)\, f_v(s)\, \mathrm{d}s \right\|_{\alpha-1} \le C_3\, N_1\,(t_2 - t_1)\,(|\log(t_2 - t_1)| + 1), \tag{3.6}$$

where $N_1 = \sup_{s\in I} \|f_v(s)\|$ and C_3 is some positive constant.

Using estimates (3.4), (3.5) and (3.6), we get from the inequality (3.3),

$$\|Fv(t_2) - Fv(t_1)\|_{\alpha-1} \leq L_\alpha |t_2 - t_1|, \tag{3.7}$$

where $L_\alpha = \max\{C_1(u_0, g(0,u_0)), \frac{\|A(0)^{\alpha-\beta-1}\| L_g}{1-\|A(0)^{\alpha-\beta-1}\| L_g L_a}, C_3 N_1(|\log(t_2-t_1)|+1)\}$ that depends on C_1, C_2, C_3, N_1, and δ.

Next our aim is to show $\sup_{t\in I} \|F(y)(t) - u_0\|_\alpha \leq r$, for sufficiently small $\delta > 0$. Since $u_0 + g(0,u_0) \in X_\alpha$, we can choose sufficiently small $\delta_1 > 0$ such that [2, Lemma II.14.1],

$$\|[U(t,0) - I]\,[u_0 + g(0,u_0)]\|_\alpha \leq \frac{r}{6}, \quad \text{for all } t \in [0,\delta_1]. \tag{3.8}$$

Making use of assumptions **(A6)** and **(A7)**, we can choose $\delta_2 > 0$ small enough such that

$$\|g(t, v(a(t))) - g(0,u_0)\|_\alpha \leq \frac{r}{6}, \quad \text{for all } t \in [0,\delta_2]. \tag{3.9}$$

Let $K_1 := \sup_{0\leq t\leq T} \|f(t,u_0,u_0)\|$. For $v \in \mathcal{S}_\alpha$, it follows from the assumption **(A4)** and Lemma 2.2 that

$$\left\|\int_0^t U(t,s)\, f_v(s)\,\mathrm{d}s\right\|_\alpha \leq N \left\|\int_0^t U(t,s)\, f_v(s)\,\mathrm{d}s\right\|_1 \leq CN\, \|f_v\|_{C^\gamma(I;X)}\,, \tag{3.10}$$

where N is the constant in the embedding $X_1 \hookrightarrow X_\alpha$. Indeed, we have the following sharp estimate [16, cf. inequality (1.65), page 23],

$$\begin{aligned}\left\|\int_0^t U(t,s)\, f_v(s)\,\mathrm{d}s\right\|_\alpha &\leq \left\|\int_0^t U(t,s)\{f(s,v(s),v([h(v(s),s)])) - f(s,u_0,u_0)\}\,\mathrm{d}s\right\|_\alpha \\ &\quad + \left\|\int_0^t U(t,s)\, f(s,u_0,u_0)\,\mathrm{d}s\right\|_\alpha .\end{aligned}$$

Now using (2.6), (2.7) and $h(u_0,0) = 0$, we get

$$\begin{aligned}&\left\|\int_0^t U(t,s)\, f_v(s)\,\mathrm{d}s\right\|_\alpha \\ &\leq C(\alpha)\, L_f \int_0^t (t-s)^{-\alpha}\left[\|v(s)-u_0\|_\alpha + \|v([h(v(s),s)]) - u_0\|_{\alpha-1}\right]\,\mathrm{d}s + C(\alpha)\, K_1 \int_0^t (t-s)^{-\alpha}\,\mathrm{d}s \\ &\leq C(\alpha) L_f \int_0^t (t-s)^{-\alpha}[\|v(s)-u_0\|_\alpha + L_\alpha |h((v(s),s)) - h(u(0),0)|\,\mathrm{d}s + \frac{C(\alpha)\, K_1 \delta^{1-\alpha}}{1-\alpha} \\ &\leq C(\alpha)\, L_f \int_0^t (t-s)^{-\alpha}[r + L_\alpha L_h(\|v(s)-u_0\|_\alpha + s^{\theta_2})]\,\mathrm{d}s + \frac{C(\alpha) K_1 \delta^{1-\alpha}}{1-\alpha} \\ &\leq C(\alpha)\, L_f[(1+L_\alpha L_h)r + \delta^{\theta_2}] \int_0^t (t-s)^{-\alpha}\,\mathrm{d}s + \frac{C(\alpha)\, K_1 \delta^{1-\alpha}}{1-\alpha} \\ &\leq \left(\frac{C(\alpha)}{1-\alpha}\, L_f\,[(1+L_\alpha\, L_h)\, r + \delta^{\theta_2}] + \frac{C(\alpha) K_1}{1-\alpha}\right) \delta^{1-\alpha}.\end{aligned} \tag{3.11}$$

We choose $\delta_3 > 0$ such that

$$\left(\frac{C(\alpha)}{1-\alpha}\, L_f\, [(1+L_\alpha\, L_h)\, r + \delta_3^{\theta_2}] + \frac{C(\alpha) K_1}{1-\alpha}\right) \delta_3^{1-\alpha} \leq \frac{2\, r}{3}.$$

Combining (3.8), (3.9) and (3.11), we obtain $\sup_{t\in I} \|Fv(t) - u_0\|_\alpha \leq r$.

Our next aim is to show that $\|Fv(t+h) - Fv(t)\|_\alpha \leq Kh^\eta$, for some $K > 0$ and $0 < \eta < 1$. If $0 \leq \alpha < \beta \leq 1, 0 \leq t \leq t+h \leq \delta$, then we have

$$\begin{aligned}\|Fv(t+h) - Fv(t)\|_\alpha \leq \; & \|[U(t+h,0) - U(t,0)]\,[u_0 + g(0,u_0)\|_\alpha \\ & + \|g(t+h, v(a(t+h))) - g(t, v(a(t)))\|_\alpha \\ & + \Big\| \int_0^{t+h} U(t+h,s)\, f_v(s)\,\mathrm{d}s - \int_0^t U(t,s)\, f_v(s)\,\mathrm{d}s \Big\|_\alpha . \end{aligned} \tag{3.12}$$

Using [2, Lemma II.14.1 and Lemma II.14.4], we get the following estimates

$$\|[U(t+h,0) - U(t,0)]\,[u_0 + g(0,u_0)]\|_\alpha \leq C\; \|u_0 + g(0,u_0)\|_\beta\; h^{\beta-\alpha}; \tag{3.13}$$

$$\Big\| \int_0^{t+h} U(t+h,s)\, f_v(s)\,\mathrm{d}s - \int_0^t U(t,s)\, f_v(s)\,\mathrm{d}s \Big\|_\alpha \leq C(\alpha)\, N_1\, h^{1-\alpha}\,(1 + |\log h|). \tag{3.14}$$

Again making use of assumptions **(A6)** and **(A7)**, we get

$$\|g(t+h, v(a(t+h))) - g(t, v(a(t)))\|_\alpha \leq C\, L_g\,(1 + L_\alpha\, L_a)\, h \tag{3.15}$$

From (3.13), (3.14) and (3.15), it is clear that

$$\begin{aligned}\|Fv(t+h) - Fv(t)\|_\alpha \leq h^\eta \big[& C\,\|u_0 + g(0,u_0)\|_\beta \delta^{\beta-\alpha-\eta} + C\, L_g\,(1 + L_\alpha\, L_a) h^{1-\eta} \\ & + C(\alpha)\, N_1\, \delta^\nu\, h^{1-\alpha-\eta-\nu}(|\log h| + 1)\big],\end{aligned}$$

for any $\nu > 0, \nu < 1 - \alpha - \eta$. Hence, for sufficiently small $\delta > 0$, we have

$$\|Fv(t+h) - Fv(t)\|_\alpha \leq K\, h^\eta$$

for some $K > 0$. Thus, we have shown that F maps $\mathcal{S}_\alpha$ into itself.

Finally, we will show that F is a contraction map. Let $v_1, v_2 \in \mathcal{S}_\alpha$. For $t \in I$, we have [16, inequality (1.65), page 23],

$$\begin{aligned}& \|Fv_1(t) - Fv_2(t)\|_\alpha \\ & \leq L_g\, \|A(0)^{-1}\|\; \|v_1(t) - v_2(t)\|_\alpha \\ & \quad + C(\alpha)\, L_f \int_0^t (t-s)^{-\alpha}\,(\|v_1(s) - v_2(s)\|_\alpha + \|v_1([h(v_1(s),s)]) - v_2([h(v_2(s),s)])\|_{\alpha-1})\,\mathrm{d}s \\ & \leq L_g\, \big\|A(0)^{-1}\big\|\; \|v_1(t) - v_2(t)\|_\alpha + C(\alpha)\, L_f (2 + L_\alpha L_h) \int_0^t (t-s)^{-\alpha}\, \|v_1(s) - v_2(s)\|_\alpha\; \mathrm{d}s \\ & \leq \big[L_g\, \|A(0)^{-1}\| + \frac{C(\alpha)}{1-\alpha}\, L_f (2 + L_\alpha L_h)\, \delta^{1-\alpha}\big] \sup_{t\in I} \|v_1(t) - v_2(t)\|_\alpha . \end{aligned} \tag{3.16}$$

Choose $\delta_4 > 0$ such that

$$L_g\|A(0)^{-1}\| + \frac{C(\alpha)}{1-\alpha}L_f(2 + L_\alpha L_h)\delta_4^{1-\alpha} < \frac{1}{2}.$$

Then, from (3.16), it is clear that F is a contraction map. Since $\mathcal{S}_\alpha$ is a complete metric space, by the Banach fixed-point theorem, there exists $u \in \mathcal{S}_\alpha$ such that $Fu = u$. From Lemma 2.2 and Theorem 5 of Sobolevskiĭ [16], it follows that $u \in C^1((0,\delta); X)$. Thus u is a solution to (1.1) on $[0,\delta]$, where $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$. □

Next we will prove the following theorem that gives the existence of a global solution to (1.1).

Theorem 3.4 *Assume that* ***(A1)-(A7)*** *hold. If there are continuous nondecreasing real valued functions $k_1(t)$, $k_2(t)$ and $k_3(t)$ such that*

$$\begin{aligned} \|f(t,x,y)\| &\le k_1(t)\,(1 + \|x\|_\alpha + \|y\|_{\alpha-1}), && (3.17)\\ |h(x,t)| &\le k_2(t)\,(1 + \|x\|_\alpha), && (3.18)\\ \|g(t,y)\|_\beta &\le k_3(t)\,(1 + \|y\|_{\alpha-1}), && (3.19)\end{aligned}$$

for all $t \ge 0$, $x \in X_\alpha$ and $y \in X_{\alpha-1}$, then the initial value problem (1.1) has a unique solution that exists for all $t \in [0,\delta]$, for each $u_0 \in W_\beta$, where $0 < \alpha < \beta \le 1$.

Proof. From Theorem 3.3 it follows that there exists a $\delta > 0$ and a unique local solution $u(t)$ on $t \in [0,\delta]$ to Problem (1.1). If

$$\|u(t)\|_\alpha \le D$$

for all $t \in [0,\delta]$ and for some constant D that is independent of t, then the solution $u(t)$ to Problem (1.1) may be continued further on the right of δ. Thus it is enough to show that $\|u(t)\|_\alpha$ is bounded as $t \uparrow T$.

Let $k_1(T)$, $k_2(T)$ and $k_3(T)$ be the supremum of $k_1(t)$, $k_2(t)$ and $k_3(t)$, $t \in [0,T]$, respectively. Now using (2.6), (2.7), (2.8), (3.17), (3.18), (3.19) and [2, inequality (II.14.12) and (II.14.14)], we get for $u(\cdot) \in X_1$ and $t \in I$,

$$\begin{aligned}\|u(t)\|_\alpha &\le \|U(t,0)\,[u_0 + g(0,u_0)]\|_\alpha \\ &\quad + \|g(t,u(a(t)))\|_\alpha + \left\|\int_0^t U(t,\tau)\,f(\tau,u(\tau),u([h(u(\tau),\tau)]))\,\mathrm{d}\tau\right\|_\alpha \\ &\le \left\|A(0)^\alpha A(t)^{-\beta}A(t)^\beta U(t,0)\,A(0)^{-\beta}\,A(0)^\beta\,[u_0 + g(0,u_0)]\right\| \\ &\quad + \left\|A(0)^{\alpha-\beta}\right\|\,k_3(T)\left(1 + \left\|A(0)^{-1}\right\|\sup_{s\in[0,t]}\|u(s)\|_\alpha\right) \\ &\quad + k_1(T)\int_0^t (t-\tau)^{-\alpha}[(1 + \|u(\tau)\|_\alpha + L_\alpha\,|h(u(\tau),\tau) - h(u_0,0)| + \|u_0\|_{\alpha-1}]\,\mathrm{d}\tau \\ &\le \left(C\,\|u_0 + g(0,u_0)\|_\beta + \left\|A(0)^{\alpha-\beta}\right\|k_3(T) + k_1(T)\;\|u_0\|_{\alpha-1}\int_0^t (t-\tau)^{-\alpha}\,\mathrm{d}\tau\right) \\ &\quad + k_3(T)\left\|A(0)^{\alpha-\beta-1}\right\|\sup_{s\in[0,t]}\|u(s)\|_\alpha \\ &\quad + k_1(T)\,[1 + L_\alpha k_2(T)]\int_0^t (t-\tau)^{-\alpha}(1 + \sup_{\varsigma\in[0,\tau]}\|u(\varsigma)\|_\alpha)\,\mathrm{d}\tau.\end{aligned}$$

Thus we have

$$\sup_{s\in[0,t]} \|u(s)\|_\alpha \leq \tilde{C}_1 + \tilde{D}_1 \int_0^t (t-\tau)^{-\alpha}(1 + \sup_{\varsigma\in[0,\tau]} \|u(\varsigma)\|_\alpha)\,\mathrm{d}\tau,$$

where

$$\tilde{C}_1 = \frac{\left(C\,\|u_0 + g(0,u_0)\|_\beta + \left\|A(0)^{\alpha-\beta}\right\|\ k_3(T) + k_1(T)\,\|u_0\|_{\alpha-1}\sup_{t\in[0,T]}\int_0^t (t-s)^{-\alpha}\,\mathrm{d}s\right)}{(1-k_3(T)\,\|A(0)^{\alpha-\beta-1}\|)},$$

$$\tilde{D}_1 = \frac{k_1(T)\,[1 + L_\alpha k_2(T)]}{(1-k_3(T)\,\|A(0)^{\alpha-\beta-1}\|)}$$

Applying Lemma 2.4, we get that $\|u(t)\|_\alpha$ is bounded as $t \uparrow T$. □

Now we will state a theorem under more regularity condition on f and u_0. Denote $D(A(0))$ by X_1. Equip this space X_1 with the graph norm

$$\|x\|_1 := (\|x\|^2 + \|A(0)x\|^2)^{\frac{1}{2}},$$

which is equivalent to the usual norm $\|A(0)x\|$ for $x \in D(A(0))$.

Let f and h be two continuous functions. Let W_1 and W be open sets in X_1 and X respectively. For each $u \in W_1$ and $u' \in W$, there are balls such that $B_1(u,r) \subset W_1$ and $B(u',r') \subset W$. We will make use of the following stronger assumptions.

$(\mathbf{A4})'$ There exist constants $L_f = L_f(t,u,u',r,r') > 0$ and $0 < \theta_1 \leq 1$, such that the nonlinear map $f : [0,T] \times W_1 \times W \to X_\alpha$ satisfies,

$$\left\|f(t,x,x') - f(s,y,y')\right\|_\alpha \leq L_f(|t-s|^{\theta_1} + \|x-y\|_1 + \|x'-y'\|), \tag{3.20}$$

for all $x,y \in B_1$, $x',y' \in B$, for all $s,t \in [0,T]$ and $\alpha \in (0,1)$.

$(\mathbf{A5})'$ There exist constants $L_h = L_h(u',t,r') > 0$ and $0 < \theta_2 \leq 1$, such that $h(\cdot,0) = 0$, $h : W_1 \times [0,T] \to [0,T]$ satisfies,

$$|h(x,t) - h(y,s)| \leq L_h(\|x-y\|_1 + |t-s|^{\theta_2}), \tag{3.21}$$

for all $x,y \in B_1$ and for all $s,t \in [0,T]$.

$(\mathbf{A6})'$ There exists constant $L_g = L_g(t,u',r') > 0$ such that the continuous function $g : [0,T] \times W \to X_1$ satisfies

$$\begin{aligned} \|g(t,x) - g(s,y)\|_1 &\leq L_g\{|t-s| + \|x-y\|\}, \text{ and} \\ 4L_g\|A(0)^{-1}\| &< 1 \end{aligned}$$

for all $x,y \in B$ and $t,s \in [0,T]$.

Then we have the following theorem.

Theorem 3.5 *Let $u_0 \in W_1$. Let the assumptions **(A1)-(A3)**, $(\mathbf{A4})' - (\mathbf{A6})'$ and **(A7)** hold. Then there exists a positive number $\delta = \delta(u_0)$ and a unique solution $u(t)$ to Problem (1.1) on the interval $[0, \delta]$ such that $u \in C_L(I; X) \cap C^1((0, \delta); X) \cap C(I; X)$, where*

$$C_L(I; X) = \{\psi \in C(I; X_1) : \|\psi(t) - \psi(s)\| \leq L|t - s|, \text{ for all } t, s \in I\},$$

for some $L > 0$. Moreover, if there are continuous nondecreasing real valued functions $k_4(t)$, $k_5(t)$ and $k_6(t)$ such that

$$\|f(t, x, y)\|_\alpha \leq k_4(t)\,(1 + \|x\|_1 + \|y\|), \text{ for } 0 < \alpha < 1, \tag{3.22}$$
$$|h(z, t)| \leq k_5(t)\,(1 + \|z\|_1), \tag{3.23}$$
$$\|g(t, y)\|_1 \leq k_6(t)\,(1 + \|y\|), \tag{3.24}$$

for all $t \in [0, T]$, $x, z \in X_1$ and $y \in X$. Then the unique solution of (1.1) exists for all $t \geq 0$.

Proof. For each $v \in C(I, B_1)$, define a map F by

$$Fv(t) = U(t, 0)\,[u_0 + g(0, u_0)] - g(t, v(a(t))) + \int_0^t U(t, s)\, f(s, v(s), v([h(v(s), s)]))\,\mathrm{d}s$$

for each $t \in I$. By Lemma 2.3, the map F from $C(I, B_1)$ into $C(I; X_1)$ is well defined. The proof of this Theorem can be obtained by the same argument as in the proof of Theorem 3.3 and Theorem 3.4. Thus, we shall omit the details of the proof. □

Remark 3.1 *In the case when $A(t)$ is a self adjoint positive definite operator in a Hilbert space X, Theorem 3.3 and Theorem 3.4 can be strengthened. Assumptions **(A1), (A2)** and **(A3)** imply that, for $0 \leq \alpha \leq 1$ and for all $s, t \in [0, T]$ [11, page 185],*

$$\|A^\alpha(t)\, A^{-\alpha}(s)\| \leq C\,\|A(t)\, A^{-1}(s)\|^\alpha \leq \widetilde{C}, \tag{3.25}$$

where $C, \widetilde{C} > 0$. Then we can prove Theorem 3.3 and Theorem 3.4 with less regularity assumption on u_0 and g.

4 Asymptotic stability of solutions

In this section, we will discuss the asymptotic stability of a solution to (1.1) in the Banach space X. The proof is based on ideas of Friedman [2] and Webb [19].

Theorem 4.1 *Suppose that the assumptions **(A1)–(A7)** hold, $u_0 \in X_\beta$ where $0 < \alpha < \beta \leq 1$ and there exists a continuous solution $u \in X_\alpha$. Suppose there exist continuous functions ϵ_1 and ϵ_2 that map $[0, \infty)$ into $[0, \infty)$, constants $k_7 > 0$ and $k_8 > 0$ such that*

$$\|f(t, u(t), u([h(u(t), t)]))\| \leq k_7(\epsilon_1(t) + \|u(t)\|_\alpha + \|u(t)\|_{\alpha-1}), \text{ for } 0 < \alpha < 1, \tag{4.1}$$
$$\|g(t, u(a(t)))\|_\beta \leq k_8(\epsilon_2(t) + \|u(t)\|_{\alpha-1}), \tag{4.2}$$

for $t \geq 0$. Then

(i) if $\epsilon_1(t)$ and $\epsilon_2(t)$ are bounded on $[0, \infty)$, then $\|u(t)\|_\alpha$ is bounded on $[0, \infty)$;

(ii) *if* $\epsilon_1(t)$ *and* $\epsilon_2(t)$ *are of* $\mathrm{O}\,(e^{\sigma t})$ *for some* $-1 < \sigma < 0$, *then* $\|u(t)\|_\alpha = \mathrm{O}\,(e^{\sigma t})$;

(iii) *if* $\epsilon_1(t)$ *and* $\epsilon_2(t)$ *are of* $\mathrm{o}\,(1)$, *then* $\|u(t)\|_\alpha = \mathrm{o}\,(1)$.

Proof. It is known [2, page 176] that there exists $0 < \theta < d$, such that

$$\|A(t)^\gamma\, U(t,0)\| \leq \frac{C}{t^\gamma}\, e^{-\theta t}, \text{ if } t > 0, \tag{4.3}$$

for any $0 \leq \gamma \leq 1$. The solution to Problem (1.1)

$$u(t) = U(t,0)\,[u_0 + g(0,u_0)] - g(t,u(a(t))) + \int_0^t U(t,s)\, f(s,u(s),u([h(u(s),s)]))\,\mathrm{d}s,$$

for $t \in I$. Now, for $t > 0$, put $\varphi(t) = e^{\theta t}\,\|u(t)\|_\alpha$. Using (4.3) in the solution of (1.1), we obtain

$$\begin{aligned}\varphi(t) &\leq C\,t^{-\alpha}\,\|u_0 + g(0,u_0)\| + \|A(0)^{\alpha-\beta-1}\|\,k_8\big(\varphi(t) + e^{\theta t}\,\epsilon_2(t)\big)\\ &\qquad + C\int_0^t e^{\theta s}\,(t-s)^{-\alpha}\,k_7\,[\epsilon_1(s) + \|u(s)\|_\alpha + \|u(s)\|_{\alpha-1}]\,\mathrm{d}s\\ &\leq C\,t^{-\alpha}\,\|u_0 + g(0,u_0)\| + \|A(0)^{\alpha-\beta-1}\|\,k_8\big(\varphi(t) + e^{\theta t}\,\epsilon_2(t)\big)\\ &\qquad + C\,k_7\int_0^t e^{\theta s}\,(t-s)^{-\alpha}\,\epsilon_1(s)\,\mathrm{d}s + C\,k_7\,(1 + \|A(0)^{-1}\|)\int_0^t (t-s)^{-\alpha}\,\varphi(s)\,\mathrm{d}s\end{aligned}$$

Consequently, we have

$$\begin{aligned}\varphi(t) &\leq \{C_0\,t^{-\alpha}\,\|u_0 + g(0,u_0)\| + C_0\,e^{\theta t}\epsilon_2(t) + C_0\int_0^t e^{\theta s}\,(t-s)^{-\alpha}\,\epsilon_1(s)\,\mathrm{d}s\}\\ &\qquad + C_0\int_0^t (t-s)^{-\alpha}\,\varphi(s)\,\mathrm{d}s,\end{aligned} \tag{4.4}$$

where $C_0 = \dfrac{\max\{C, C\,k_7, C\,k_7\,(1 + \|A(0)^{-1}\|), 1\}}{(1 - \|A(0)^{\alpha-\beta-1}\|\,k_8)}$. Denote

$$\chi(t) = C_0\,t^{-\alpha}\,\|u_0 + g(0,u_0)\| + C_0\,e^{\theta t}\epsilon_2(t) + C_0\int_0^t e^{\theta s}\,(t-s)^{-\alpha}\,\epsilon_1(s)\,\mathrm{d}s.$$

Then it is clear that

$$\chi(t) \leq C_0\,t^{-\alpha}\,\|u_0 + g(0,u_0)\| + C_0\,e^{\theta t}\,\epsilon_2(t) + \tilde{C}\,e^{\theta t}\sup_{0\leq s<\infty}\epsilon_1(s), \tag{4.5}$$

for some constant $\tilde{C} > 0$. By the method of iteration, we get from (4.4) that

$$\varphi(t) \leq \chi(t) + \int_0^t \left[\sum_0^\infty \frac{(t-s)^{j-1-j\alpha}\,[\Gamma(1-\alpha)]^j}{\Gamma(j - j\alpha)}\right]\chi(s)\,\mathrm{d}s.$$

Since the series in the bracket is bounded by $B_1\,(t-s)^{-\alpha}\exp[B_2\,(t-s)^{1-\alpha}]$ for some constants $B_1, B_2 > 0$, it follows that, for $t \geq 1$ and for any $\lambda > 0$,

$$\varphi(t) \leq B_3\,e^{\lambda t}\,\|u_0 + g(0,u_0)\| + B_4\,e^{\theta t}\,\epsilon_2(t) + B_5\,e^{\theta t}\sup_{0\leq s<\infty}\epsilon_1(s). \tag{4.6}$$

where B_3, B_4 and B_5 are some positive constants. Thus, for any $0 < \theta_0 < \theta$, we get

$$\|u(t)\|_\alpha \leq B_3\,e^{-\theta_0 t}\,\|u_0 + g(0,u_0)\| + B_4\,\epsilon_2(t) + B_5\sup_{0\leq s<\infty}\epsilon_1(s). \tag{4.7}$$

Now the theorem follows from the inequality (4.7). □

5 Example

Example 5.1 Consider the following differential equation with deviated argument,

$$\left.\begin{aligned} \partial_t[u(t,x)+\partial_x f_1(t,u(a(t),x))] & \\ -\,\partial_x(k(t,x)\partial_x)[u(t,x)+\partial_x f_1(t,u(a(t),x))]) & \\ = \widetilde{H}(x,u(t,x))+\widetilde{G}(t,x,u(t,x)); & \\ u(t,0) = u(t,1),\ \ t>0; & \\ u(0,x) = u_0(x),\ \ x\in(0,1). & \end{aligned}\right\} \tag{5.1}$$

Here $\widetilde{H}(x,u(t,x)) = \int_0^x K(x,y)\, u(\widetilde{g}(t)|u(t,y)|,y)\,\mathrm{d}y$ for all $(t,x)\in(0,\infty)\times(0,1)$. Assume that $\widetilde{g}:\mathbb{R}_+\to\mathbb{R}_+$ is locally Hölder continuous in t with $\widetilde{g}(0)=0$ and $K\in C^1([0,1]\times[0,1];\mathbb{R})$. The function $\widetilde{G}:\mathbb{R}_+\times[0,1]\times\mathbb{R}\to\mathbb{R}$ is measurable in x, locally Hölder continuous in t, locally Lipschitz continuous in u, uniformly in x.

Suppose that k is positive function and has continuous partial derivative k_x such that, for all $0\le t<\infty$ and $x\in(0,1)$,

(i) $0<k_0\le k(t,x)<k_0^{'}$,

(ii) $|k_x(t,x)|\le k_1$,

(iii) $|k(t,x)-k(s,x)|\le C|t-s|^{\epsilon}$,

(iv) $|k_x(t,x)-k_x(s,x)|\le C|t-s|^{\epsilon}$,

for some ϵ with $0<\epsilon\le 1$, some constants k_0, $k_0^{'}$, and $C>0$. Let $X=L^2((0,1);\mathbb{R})$, $A(t)u(t)=-\frac{\partial}{\partial x}(k(t,x)\frac{\partial}{\partial x}u(x))$. Then $X_1=D(A(0))=H^2(0,1)\cap H_0^1(0,1)$ and $X_{1/2}=D((A(0))^{1/2})=H_0^1(0,1)$. It is standard that the family $\{A(t):t>0\}$ satisfy the assumptions **(A1)**, **(A2)** and **(A3)** on each bounded interval $[0,T]$.

For $x\in(0,1)$, we define $f:\mathbb{R}_+\times H^2(0,1)\times L^2(0,1)\to H_0^1(0,1)$ by

$$f(t,\phi,\psi)=\widetilde{H}(x,\psi)+\widetilde{G}(t,x,\phi),$$

where $\widetilde{H}(x,\psi(x,t))=\int_0^x K(x,y)\psi(y,t)\,\mathrm{d}y$ and $\widetilde{G}:\mathbb{R}_+\times[0,1]\times H^2(0,1)\to H_0^1(0,1)$ satisfies $\|\widetilde{G}(t,x,u)\|_{H_0^1(0,1)}\le C(1+\|u\|_{H^2(0,1)})$, for some $C>0$.

Then it can seen that f satisfies the condition (3.20) (see Gal [4]) and $h:H^2(0,1)\times\mathbb{R}_+\to\mathbb{R}_+$ defined by $h(\phi(x,t),t)=\widetilde{g}(t)|\phi(x,t)|$ satisfies (3.21) (see Gal [4]).
We also assume that the function $g:\mathbb{R}_+\times H_0^1(0,1)\to L^2(0,1)$, such that

$$g(t,u(a(t)))(x)=\partial_x f_1(t,u(a(t),x)),$$

satisfies the assumption **(A6)**. We can take the function a to be one of the following:

(i) $a(t)=k\,t$ for $t\in[0,T]$ and $0<k\le 1$;

(ii) $a(t) = k\,t^n$ for $t \in [0,1]$, $k \in (0,1]$ and $n \in \mathbb{N}$;

(iii) $a(t) = k \sin t$ for $t \in [0, \frac{\pi}{2}]$ and $k \in (0,1]$.

Thus, we can apply the results of previous sections to study the existence, uniqueness and asymptotic stability of solution to (5.1).

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