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# PROPERTIES OF THE METHOD OF SUCCESSIVE APPROXIMATIONS FOR TWO-POINT BOUNDARY VALUE PROBLEMS

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**Abstract.** The convergence and numerical stability properties of the method of successive approximations applied to two-point boundary value problems for nonlinear second order neutral differential equations in Banach spaces are obtained. The constructed algorithm has a practical stopping criterion and its accuracy is illustrated on three numerical examples.

**Keywords:** Two-point boundary value problem; fixed point technique; approximation of the solution.

**2010 AMS Mathematics Subject Classification:** 34B15.

## 1 Introduction

Consider the following two-point boundary value problem:

$$\begin{cases} y''(x) = f\left(x, y(x), y'(x)\right), \qquad x \in [a,b] \\ y(a) = c, \quad y\left(b\right) = d. \end{cases} \tag{1.1}$$

The existence and uniqueness of the solution of (1.1) on the real axis is studied in [4] and [29] using the Perov's fixed point theorem (see [29]). The framework of this fixed point theorem are generalized metric spaces with vector-valued metric.

It is known that the two-point boundary value problem (1.1) is equivalent to the following integro-differential equation :

$$y(x) = \frac{x-a}{b-a} \cdot d + \frac{b-x}{b-a} \cdot c - \int_a^b G(x,s)\, f(s, y(s), y'(s))\, \mathrm{d}s, \quad x \in [a,b] \tag{1.2}$$

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where

$$G(x,s)=\begin{cases}\frac{(s-a)(b-x)}{b-a} & \text{if } s\le x,\\ \frac{(x-a)(b-s)}{b-a} & \text{if } s\ge x\end{cases}$$

is the well-known Green's function. Differentiating the equation (1.2) we get

$$y'(x)=\frac{d-c}{b-a}-\int_a^b\frac{\partial G}{\partial x}\cdot f\left(s,y(s),y'(s)\right)\,\mathrm{d}s,\quad x\in[a,b]$$

with

$$\frac{\partial G}{\partial x}(x,s)=\begin{cases}-\frac{(s-a)}{b-a} & \text{if } s<x,\\ \frac{(b-s)}{b-a} & \text{if } s>x.\end{cases}$$

In what follows we denote $\frac{1}{b-a}\cdot(d-c)=\frac{d-c}{b-a}$. Denoting $z=y'$ we obtain the following system of Fredholm integral equations

$$\begin{cases}y(x)=\frac{x-a}{b-a}\cdot d+\frac{b-x}{b-a}\cdot c-\int\limits_a^b G(x,s)\cdot f\left(s,y(s),z(s)\right)\,\mathrm{d}s,\\ z(x)=\frac{d-c}{b-a}-\int\limits_a^b\frac{\partial G}{\partial x}(x,s)\cdot f\left(s,y\left(s\right),z(s)\right)\,\mathrm{d}s,\end{cases}\quad x\in[a,b].\tag{1.3}$$

To this system (1.3) is applied the Perov's fixed point theorem.

It is well-known that the method of successive approximations was firstly applied to differential equations by Picard and Lindelöf (see [19], [30]). For instance, the existence and uniqueness of the solution of the boundary value problem

$$\begin{cases}y''(x)=f\left(x,y(x)\right), & x\in[a,b],\\ y(a)=c,\quad y(b)=d, & c,d\in\mathbb{R}.\end{cases}\tag{1.4}$$

was proved by Picard using the method of successive approximations to the integral equation

$$y(x)=\frac{x-a}{b-a}\cdot c+\frac{b-x}{b-a}\cdot d-\int_a^b G(x,s)\cdot f\left(s,y(s)\right)\,\mathrm{d}s,\quad x\in[a,b].\tag{1.5}$$

The study of the periodic solutions of two-point boundary value problems using the method of successive approximations can be found in [36] and [39]. In this paper we apply the Perov's fixed point theorem to the boundary value problem (1.1) in Banach spaces to obtain the existence, uniqueness and approximation of the solution. The approximation scheme is based on the technique of successive approximations obtaining a convergent and stable algorithm.

In the last sixty years many numerical methods were developed for two-point boundary value problems. The involved techniques are based on Nyström methods (see [25]), shooting methods (see [1], [24], [11], [12], [15], [20]), finite differences methods (see [11], [31], [40]), Rayleigh-Ritz methods (see [28]), Galerkin methods (see [1], [3], [10]), interpolation techniques (see [23]), extrapolation schemes (see [42]), iterative methods (see [43]), projection methods (see [33], [34]), Adomian decomposition methods (see [14]), Taylor series methods (see [13]), spline functions method (see [8], [17], [18], [22], [41]) and collocation methods (see [2], [26], [32], [35] and [21] for Hammerstein integral equations which generalize the equation (1.5)). The method of successive approximations is applied to the boundary value problem (1.4) in [16] and [27]. In the present paper we construct an approximation scheme for the solution of the boundary value problem (1.1) based on the method of successive approximations.

The paper is organized as follows. Section 2 presents the notion of generalized metric space and the Perov's fixed point theorem. The properties of the sequence of successive approximations associated to the boundary value problem (1.1) are obtained in Section 3. The approximation method and the corresponding algorithm are developed in Section 4. The convergence of the method and the stability properties of the approximate solution are proved in Section 5. Some concluding remarks about the accuracy of the method are presented in Section 6.

## 2 Generalized metrics and the Perov's fixed point theorem

Let $X \neq \emptyset$, $n \in \mathbb{N}$ and $d : X \times X \to \mathbb{R}^n_+$ where,

$$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, \quad \forall i = \overline{1, n}\}.$$

**Definition 2.1** *(see [4]) The pair $(X, d)$ is a generalized metric space iff the function $d$ has the following properties:*
*(gm1) $d(x, y) \geq 0$, $\forall x, y \in X$ and $d(x, y) = 0 \Longleftrightarrow x = y$,*
*(gm2) $d(y, x) = d(x, y)$, $\forall x, y \in X$,*
*(gm3) $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in X$.*
*The function $d$ is then called a generalized metric.*

The euclidean space $\mathbb{R}^n$ is ordered by the relation :

$$x \leq y \Longleftrightarrow x_i \leq y_i, \quad \forall i = \overline{1, n},$$

for $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$.

A generalized metric space is complete if any fundamental sequence in $X$ is convergent. Let $M_n(\mathbb{R}_+)$ the set of matrices with positive elements.

**Definition 2.2** *(see [4]) Let $(X, d)$ be a generalized metric space. A map $T : X \to X$ satisfy a generalized Lipschitz inequality if there exists a matrix $A \in M_n(\mathbb{R}_+)$ such that :*

$$d(T(x), T(y)) \leq A d(x, y), \quad \forall x, y \in X.$$

**Remark 2.3** *In matrix calculus, for a matrix $A \in M_n(\mathbb{R}_+)$, the following properties are equivalent:*

*(i) $A^m \to 0$ as $m \to \infty$,*

*(ii) all eigenvalues of $A$ lie in the open unit ball of the complex plane,*

*(iii) the matrix $(I_n - A)$ is invertible and*

$$(I_n - A)^{-1} = I + A + A^2 + \dots + A^n + \dots$$

**Theorem 2.4** *(see [29], [4]). Let $(X, d)$ be a generalized metric space and $A : X \to X$ a mapping which has the generalized Lipschitz inequality property with a matrix $Q \in M_n(\mathbb{R}_+)$. If all eigenvalues of $Q$ lie in the open unit ball of the complex plane, then:*

*(i) The operator $A$ has a unique fixed point $x^* \in X$.*

*(ii)* *For any $x_0 \in X$, the sequence of successive approximations $(x_m)_{m\in\mathbb{N}} \subset X$ defined by $x_m = A(x_{m-1})$, $\forall m \in \mathbb{N}^*$, is convergent to $x^*$.*

*(iii)* *The following inequalities hold:*

$$d(x_m, x^*) \le Q^m \cdot (I_n - Q)^{-1} \cdot d(x_0, x_1), \qquad \forall m \in \mathbb{N}^*, \tag{2.1}$$

$$d(x_m, x^*) \le Q \cdot (I_n - Q)^{-1} \cdot d(x_m, x_{m-1}), \qquad \forall m \in \mathbb{N}^*. \tag{2.2}$$

As an application of the Perov's fixed point theorem, I. A. Rus obtains the result:

**Theorem 2.5** *(of fiber generalized contractions, Rus [37], [38]). Let $(X, d)$ be a metric space (generalized or not) and $(Y, \rho)$ be a complete generalized metric space ( $\rho(x, y) \in \mathbb{R}^n_+$ ). Let $A : X \times Y \to X \times Y$ be a continuous operator and $C : X \times Y \to Y$ an operator. Suppose that :*

*(i)* *$B : X \to X$ has an unique fixed point as limit of the sequence of successive approximations associated to $B$,*

*(ii)* *$A(x, y) = (B(x), C(x, y))$, for all $x \in X$, $y \in Y$,*

*(iii)* *there exists a matrix $Q \in M_n(\mathbb{R}_+)$, with $Q^m \to 0$ as $m \to \infty$, such that*

$$\rho(C(x, y_1), C(x, y_2)) \le Q \cdot \rho(y_1, y_2),$$

*for all $x \in X$, $y_1$ and $y_2 \in Y$. Then, the operator $A$ has an unique fixed point as limit of the sequence of successive approximations associated to this operator.*

This theorem of fiber generalized contractions is applied to obtain the smooth dependence by parameters of the solution of operatorial equations (see [38]). The same theorem is applied in [6] proving the smooth dependence of the solution of (1.1) by the end points $a$ and $b$.

# 3 The method of successive approximations

Let $X$ be a real Banach space. The following notations will be used:

$$C([a, b], X) = \{f : [a, b] \to X \mid f \text{ is continuous on } [a, b]\}$$

$$C^k([a, b], X) = \{f : [a, b] \to X \mid f \text{ is } k \text{ times differentiable on } [a, b] \text{ with } f^{(k)} \text{ continuous on } [a, b]\},$$

$$Lip([a, b], X) = \{f : [a, b] \to X \mid f \text{ is Lipschitzian on } [a, b]\}.$$

Remember that $f : [a, b] \to X$ is Lipschitzian on $[a, b]$ iff there exists $L \ge 0$ such that

$$\left\|f(x) - f(x')\right\|_X \le L \cdot \left|x - x'\right|, \text{ for any } x, x' \in [a, b]$$

and $f : [a, b] \to X$ is bounded on $[a, b]$ iff there exists $M \ge 0$ such that $\|f(x)\|_X \le M$ for all $x \in [a, b]$. According to the boundary value problem (1.1) we consider the following conditions:

(i) $f \in C([a, b] \times X \times X, X)$, and $c, d \in X$,

(ii) there exist $\alpha, \beta \ge 0$ such that

$$\left\|f(x, u, v) - f(x, u', v')\right\|_X \le \alpha \cdot \left\|u - u'\right\|_X + \beta \cdot \left\|v - v'\right\|_X$$

for any $u, v, u', v' \in X$,

(iii) $\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a) < 1$.

To prove the convergence of the method we impose the following supplementary condition:

(iv) There exist $\gamma \geq 0$ such that

$$\left\| f(x,u,v) - f(x',u,v) \right\|_X \leq \gamma \cdot \left| x - x' \right|, \text{ for any } x, x' \in [a,b].$$

Consider the generalized metric $d_C : X \times X \to \mathbb{R}^2$, defined by

$$d_C((y_1,z_1),(y_2,z_2)) = (\|y_1 - y_2\|_C \ , \ \|z_1 - z_2\|_C), \quad \forall (y_1,z_1),(y_2,z_2) \in Y$$

where $Y = C([a,b],X) \times C([a,b],X)$ and

$$\|y\|_C = \max\{\|y(x)\|_X : x \in [a,b]\} \text{ for } y \in C([a,b],X).$$

Let $g, h : [a,b] \to X$, $g(x) = \frac{x-a}{b-a} \cdot d + \frac{b-x}{b-a} \cdot c$, $h(x) = \frac{d-c}{b-a}$, $\quad x \in [a,b]$. We see that

$$\|g\|_C \leq \max\left(\|c\|_X, \|d\|_X\right) \overset{notation}{=} r, \quad \|h\|_C \leq \frac{\|c\|_X + \|d\|_X}{b-a} \overset{notation}{=} q$$

and since $f \in C([a,b] \times X \times X, X)$ we infer that the function $f_0 : [a,b] \to X$

$$f_0(s) = f(s, g(s), h(s)), \quad s \in [a,b]$$

is bounded having $\|f_0(s)\|_X \leq M_0$ for all $s \in [a,b]$, where

$$M_0 = \max\{\|f(s,u,v)\|_X : s \in [a,b], u \in B_r, v \in B_q\}$$

and $B_r = \{u \in X : \|u\|_X \leq r\}$, $B_q = \{u \in X : \|u\|_X \leq q\}$.

**Theorem 3.1** *Under the conditions (i)-(iii), the boundary value problem (1.1) has an unique solution in $y^* \in C^2([a,b],X)$ and the sequence of successive approximations $(y_m, z_m)_{m \in \mathbb{N}} \subset C([a,b],X) \times C([a,b],X)$ given by*

$$y_0(x) = \frac{x-a}{b-a} \cdot d + \frac{b-x}{b-a} \cdot c, \quad z_0(x) = \frac{d-c}{b-a}, \quad x \in [a,b]$$

$$y_m(x) = \frac{x-a}{b-a} \cdot d + \frac{b-x}{b-a} \cdot c - \int_a^b G(x,s) \cdot f(s, y_{m-1}(s), z_{m-1}(s)) \, \mathrm{d}s, \quad m \in \mathbb{N}^* \tag{3.1}$$

$$z_m(x) = (y_m)'(x) = \frac{d-c}{b-a} - \int_a^b \frac{\partial G}{\partial x}(x,s) \cdot f(s, y_{m-1}(s), z_{m-1}(s)) \, \mathrm{d}s, \quad m \in \mathbb{N}^* \tag{3.2}$$

*has the following properties:*

*(i)* $\lim\limits_{m \to \infty} y_m(x) = y^*(x)$ *and* $\lim\limits_{m \to \infty} z_m(x) = (y^*)'(x)$ *uniformly in* $C([a,b],X)$.

*(ii) The following estimation holds: For all $m \in \mathbb{N}^*$,*

$$d_C((y_m, z_m), (y^*, (y^*)')) \le \frac{\left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]^{m-1}}{1 - \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]} \cdot Q \cdot \begin{pmatrix} (b-a)^2 \cdot \frac{M_0}{8} \\ (b-a) \cdot \frac{M_0}{2} \end{pmatrix}, \tag{3.3}$$

$$d_C((y_m, z_m), (y^*, (y^*)')) \le \frac{\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)}{1 - \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]} \cdot Q \cdot d_C((y_m, z_m), (y_{m-1}, z_{m-1})), \tag{3.4}$$

*where*

$$Q = \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix}. \tag{3.5}$$

*(iii) The terms of the sequence of successive approximations are uniformly bounded.*

*Proof.* For the case $X = \mathbb{R}$, the properties (i) and (ii) are proved in [29], Theorem 7, page 256. For the case of arbitrary Banach space $X$ we define the operator $A : Y \to Y$, $A = (A_1, A_2)$,

$$A_1(u, v)(x) = \frac{x-a}{b-a} \cdot d + \frac{b-x}{b-a} \cdot c - \int_a^b G(x, s) \cdot f(s, u(s), v(s))\, \mathrm{d}s$$

$$A_2(u, v)(x) = \frac{d-c}{b-a} - \int_a^b \frac{\partial G}{\partial x}(x, s) \cdot f(s, u(s), v(s))\, \mathrm{d}s$$

and using the fixed point technique, after elementary calculus, we get

$$\|A_1(u_1, v_1) - A_1(u_2, v_2)\|_C \le \frac{\alpha}{8}(b-a)^2 \cdot \|u_1 - u_2\|_C + \frac{\beta}{8}(b-a)^2 \cdot \|v_1 - v_2\|_C$$

and

$$\|A_2(u_1, v_1) - A_2(u_2, v_2)\|_C \le \frac{\alpha}{2}(b-a) \cdot \|u_1 - u_2\|_C + \frac{\beta}{2}(b-a) \cdot \|v_1 - v_2\|_C$$

for any $(u_1, v_1), (u_2, v_2) \in Y$. So,

$$d_C(A(u_1, v_1), A(u_2, v_2)) \le \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix} \cdot d_C((u_1, v_1), (u_2, v_2))$$

and the eigenvalues of $Q$ are $\lambda_1 = 0$, $\lambda_2 = \frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a) < 1$. Then, $A$ is contraction having an unique fixed point $(y^*, z^*) \in Y$, that is

$$\begin{cases} y^*(x) = \frac{x-a}{b-a} \cdot d + \frac{b-x}{b-a} \cdot c - \int_a^b G(x, s) \cdot f(s, y^*(s), z^*(s))\, \mathrm{d}s \\ z^*(x) = \frac{d-c}{b-a} - \int_a^b \frac{\partial G}{\partial x}(x, s) \cdot f(s, y^*(s), z^*(s))\, \mathrm{d}s, \end{cases} \quad \forall x \in [a, b]. \tag{3.6}$$

Differentiating in (3.6) we obtain $y^* \in C^2([a, b], X)$ and $z^* = (y^*)'$. Applying the Theorem 2.4, we obtain

$$d_C((y_m, z_m), (y^*, (y^*)')) \le \frac{\left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]^{m-1}}{1 - \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]} \cdot Q \cdot d_C((y_1, z_1), (y_0, z_0)), \quad \forall m \in \mathbb{N}^*$$

and since

$$d_C((y_1,z_1),(y_0,z_0)) \le \begin{pmatrix} \int_a^b |G(x,s)| \cdot \|f_0(s)\|_X \, ds \\ \int_a^b \left|\frac{\partial G}{\partial x}(x,s)\right| \cdot \|f_0(s)\|_X \, ds \end{pmatrix} \le \begin{pmatrix} (b-a)^2 \cdot \frac{M_0}{8} \\ (b-a) \cdot \frac{M_0}{2} \end{pmatrix},$$

the properties (i) and (ii) follows and $y^*$ is the unique solution of the boundary value problem (1.1). To prove the boundedness of the terms of the sequence of successive approximations we observe that

$$\begin{aligned} \|y_m(x) - y_{m-1}(x)\|_X \le & \int_a^b |G(x,s)| \cdot (\alpha \|y_{m-1} - y_{m-2}\|_C + \beta \|z_{m-1} - z_{m-2}\|_C) \, ds \\ \le & \frac{\alpha}{8}(b-a)^2 \cdot \|y_{m-1} - y_{m-2}\|_C + \frac{\beta}{8}(b-a)^2 \cdot \|z_{m-1} - z_{m-2}\|_C \end{aligned}$$

and

$$\begin{aligned} \|z_m(x) - z_{m-1}(x)\|_X \le & \int_a^b \left|\frac{\partial G}{\partial x}(x,s)\right| \cdot (\alpha \|y_{m-1} - y_{m-2}\|_C + \beta \|z_{m-1} - z_{m-2}\|_C) \, ds \\ \le & \frac{\alpha}{2}(b-a) \cdot \|y_{m-1} - y_{m-2}\|_C + \frac{\beta}{2}(b-a) \cdot \|z_{m-1} - z_{m-2}\|_C \end{aligned}$$

for any $x \in [a,b]$ and consequently, by induction we obtain:

$$\begin{aligned} & \begin{pmatrix} \|y_m - y_{m-1}\|_C \\ \|z_m - z_{m-1}\|_C \end{pmatrix} \le \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix} \cdot \begin{pmatrix} \|y_{m-1} - y_{m-2}\|_C \\ \|z_{m-1} - z_{m-2}\|_C \end{pmatrix} \\ & \le \cdots \le \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix}^{m-1} \cdot \begin{pmatrix} \|y_1 - y_0\|_C \\ \|z_1 - z_0\|_C \end{pmatrix} \\ & \le \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]^{m-2} \cdot \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix} \cdot \begin{pmatrix} (b-a)^2 \cdot \frac{M_0}{8} \\ (b-a) \cdot \frac{M_0}{2} \end{pmatrix}. \end{aligned}$$

Then,

$$\begin{aligned} \begin{pmatrix} \|y_m - y_1\|_C \\ \|z_m - z_1\|_C \end{pmatrix} \le & \begin{pmatrix} \|y_m - y_{m-1}\|_C \\ \|z_m - z_{m-1}\|_C \end{pmatrix} + \begin{pmatrix} \|y_{m-1} - y_{m-2}\|_C \\ \|z_{m-1} - z_{m-2}\|_C \end{pmatrix} + \cdots + \begin{pmatrix} \|y_2 - y_1\|_C \\ \|z_2 - z_1\|_C \end{pmatrix} \\ \le & \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix}^{m-1} \cdot \begin{pmatrix} \|y_1 - y_0\|_C \\ \|z_1 - z_0\|_C \end{pmatrix} \\ & + \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix}^{m-2} \cdot \begin{pmatrix} \|y_1 - y_0\|_C \\ \|z_1 - z_0\|_C \end{pmatrix} \\ & + \cdots + \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix} \cdot \begin{pmatrix} \|y_1 - y_0\|_C \\ \|z_1 - z_0\|_C \end{pmatrix} \\ \le & \ (Q^{m-1} + Q^{m-2} + \cdots + Q) \cdot \begin{pmatrix} \|y_1 - y_0\|_C \\ \|z_1 - z_0\|_C \end{pmatrix} \end{aligned}$$

$$\leq \left( \left[ \frac{\alpha(b-a)^2}{8} + \frac{\beta(b-a)}{2} \right]^{m-2} + \left[ \frac{\alpha(b-a)^2}{8} + \frac{\beta(b-a)}{2} \right]^{m-3} + \cdots + 1 \right) \cdot Q \cdot \begin{pmatrix} \|y_1 - y_0\|_C \\ \|z_1 - z_0\|_C \end{pmatrix}$$

$$\leq \frac{1 - \left[ \frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a) \right]^{m-1}}{1 - \left[ \frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a) \right]} \cdot Q \cdot \begin{pmatrix} \|y_1 - y_0\|_C \\ \|z_1 - z_0\|_C \end{pmatrix}$$

$$\leq \frac{1}{1 - \left[ \frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a) \right]} \cdot Q \cdot \begin{pmatrix} (b-a)^2 \cdot \frac{M_0}{8} \\ (b-a) \cdot \frac{M_0}{2} \end{pmatrix} \overset{notation}{=} \begin{pmatrix} M_1 \\ M_1' \end{pmatrix}.$$

So,

$$\begin{pmatrix} \|y_m - y_0\|_C \\ \|z_m - z_0\|_C \end{pmatrix} \leq \begin{pmatrix} M_1 \\ M_1' \end{pmatrix} + \begin{pmatrix} (b-a)^2 \cdot \frac{M_0}{8} \\ (b-a) \cdot \frac{M_0}{2} \end{pmatrix}$$

and

$$\begin{pmatrix} \|y_m\|_C \\ \|z_m\|_C \end{pmatrix} \leq \begin{pmatrix} \|y_m - y_0\|_C \\ \|z_m - z_0\|_C \end{pmatrix} + \begin{pmatrix} \|y_0\|_C \\ \|z_0\|_C \end{pmatrix}$$
$$\leq \begin{pmatrix} M_1 \\ M_1' \end{pmatrix} + \begin{pmatrix} (b-a)^2 \cdot \frac{M_0}{8} \\ (b-a) \cdot \frac{M_0}{2} \end{pmatrix} + \begin{pmatrix} r \\ q \end{pmatrix} \overset{notation}{=} \begin{pmatrix} R \\ R' \end{pmatrix} \tag{3.7}$$

for any $m \in \mathbb{N}^*$. □

**Remark 3.2** *The same conditions as in the previous theorem lead in the case $X = \mathbb{R}$ to the existence and uniqueness Theorem 7, page 256 in [29].*

We define the functions $F_m : [a,b] \to X$,

$$F_m(s) = f\left(s, y_m(s), z_m(s)\right), \quad \forall s \in [a,b], \quad \forall m \in \mathbb{N}.$$

**Corollary 1** *The solution of the boundary value problem (1.1) and its first and second derivative are bounded. The functions $F_m$, $m \in \mathbb{N}$, are uniformly bounded.*

*Proof.* Passing to limit for $m \to \infty$ in the inequality

$$\begin{pmatrix} \|y^*\|_C \\ \left\|(y^*)'\right\|_C \end{pmatrix} \leq \begin{pmatrix} \|y^* - y_m\|_C \\ \left\|(y^*)' - z_m\right\|_C \end{pmatrix} + \begin{pmatrix} \|y_m\|_C \\ \|z_m\|_C \end{pmatrix}$$
$$\leq \frac{\left[ \frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a) \right]^{m-1}}{1 - \left[ \frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a) \right]} \cdot Q \cdot \begin{pmatrix} (b-a)^2 \cdot \frac{M_0}{8} \\ (b-a) \cdot \frac{M_0}{2} \end{pmatrix} + \begin{pmatrix} R \\ R' \end{pmatrix}, \quad \forall m \in \mathbb{N}^*,$$

we get $\|y^*\|_C \leq R$ and $\left\|(y^*)'\right\|_C \leq R'$. Since the functions $F_m$, $m \in \mathbb{N}$, are continuous, it follows that there exists $M \geq 0$ such that

$$\|F_m\|_C \leq M = \max\{\|f(s,u,v)\|_X : s \in [a,b], u \in B(0,R), v \in B\left(0,R'\right)\}, \quad \forall m \in \mathbb{N},$$

where $B(0,R) = \{u \in X : \|u\|_X \leq R\}$ and $B(0,R') = \{v \in X : \|v\|_X \leq R'\}$.
Finally, since

$$(y^*)''(x) = f\left(x, y^*(x), (y^*)'(x)\right)$$

and

$$y_m''(x) = f\left(x, y_{m-1}(x), y_{m-1}'(x)\right), \quad \forall m \in \mathbb{N}^*,$$

we get $\|y_m''\|_C \leq M$ for all $m \in \mathbb{N}$ and $\left\|(y^*)''\right\|_C \leq M$. Moreover,

$$\left\|(y^*)'' - y_m''\right\|_C \leq \left( \frac{\left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]^{m-1}}{1 - \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]} \right) \cdot \left[ \alpha \left( \frac{\alpha}{64}(b-a)^4 + \frac{\beta}{16}(b-a)^3 \right) \right.$$
$$\left. + \beta \left( \frac{\alpha}{16}(b-a)^3 + \frac{\beta}{4}(b-a)^2 \right) \right] M_0, \quad \forall m \in \mathbb{N}^*. \tag{3.8}$$

□

**Remark 3.3** *From the inequalities (3.3) and (3.8) it follows that* $\lim_{m\to\infty} y_m(x) = y^*(x)$, $\lim_{m\to\infty} z_m(x) = (y^*)'(x)$, $\lim_{m\to\infty} y_m''(x) = (y^*)''(x)$, $\forall x \in [a,b]$, *uniformly in* $C([a,b], X)$. *So, the terms of the sequence of successive approximations given in (3.1) and (3.2) approximate the solution and its first derivative.*

In order to compute the integrals from (3.1) and (3.2) we apply a quadrature rule considering an uniform partition of the interval $[a,b]$ :

$$\Delta : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \tag{3.9}$$

with

$$x_i = a + \frac{i \cdot (b-a)}{n}, \quad i = \overline{0,n}.$$

On these knots the relations (3.1) and (3.2) can be written as follows:

$$y_m(x_i) = \frac{x_i - a}{b-a} \cdot d + \frac{b - x_i}{b-a} \cdot c - \int_a^b G(x_i, s) \cdot f(s, y_{m-1}(s), z_{m-1}(s))\, \mathrm{d}s, \quad i = \overline{0,n}, \tag{3.10}$$

$$z_m(x_i) = \frac{d-c}{b-a} - \int_a^b \frac{\partial G}{\partial x}(x_i, s) \cdot f(s, y_{m-1}(s), z_{m-1}(s))\, \mathrm{d}s, \quad i = \overline{0,n}. \tag{3.11}$$

Define the functions $H_{m,i}, K_{m,i} : [a,b] \to X$,

$$H_{m,i}(s) = G(x_i, s) \cdot F_m(s) = G(x_i, s) \cdot f(s, y_m(s), z_m(s)), \forall s \in [a,b],\ \forall m \in \mathbb{N},\ i = \overline{0,n},$$

$$K_{m,i}(s) = \frac{\partial G}{\partial x}(x_i, s) \cdot F_m(s) = \frac{\partial G}{\partial x}(x_i, s) \cdot f(s, y_m(s), z_m(s)), \forall s \in [a,b],\ \forall m \in \mathbb{N},\ i = \overline{0,n}.$$

**Proposition 3.4** *Under the conditions (i)-(iv), the functions* $F_m$, $m \in \mathbb{N}$, *are uniformly Lipschitz with the Lipschitz constant*

$$L_0 = \gamma + \alpha \left( \frac{\|c\|_X + \|d\|_X}{b-a} + \frac{M(b-a)^2}{2} \right) + \beta M$$

*and the functions* $H_{m,i}$, $K_{m,i}$, $m \in \mathbb{N}$, $i = \overline{0,n}$ *are Lipschitzian with the same Lipschitz constant (uniformly Lipschitz)*

$$L_1 = M + \frac{(b-a)}{4} \cdot \left[ \gamma + \alpha \left( \frac{\|c\|_X + \|d\|_X}{b-a} + \frac{M(b-a)^2}{2} \right) + \beta M \right]$$

*and*

$$L_2 = \frac{M}{b-a} + \gamma + \alpha \left( \frac{\|c\|_X + \|d\|_X}{b-a} + \frac{M\,(b-a)^2}{2} \right) + \beta M,$$

*respectively.*

*Proof.* Firstly, we can see that

$$\begin{aligned}
\left\| y_m(x) - y_m\left(x'\right) \right\|_X &\leq \left\| y_m' \right\|_C \cdot \left| x - x' \right| \\
&\leq \left( \frac{\|c\|_X + \|d\|_X}{b-a} + \frac{M\,(b-a)^2}{2} \right) \cdot \left| x - x' \right| = \delta \cdot \left| x - x' \right| \\
\left\| z_m(x) - z_m\left(x'\right) \right\|_X &= \left\| y_m'(x) - y_m'\left(x'\right) \right\|_X \leq \left\| y_m'' \right\|_C \cdot \left| x - x' \right| \leq M \cdot \left| x - x' \right|, \\
\left\| F_0(x) - F_0\left(x'\right) \right\|_X &\leq \left( \gamma + \alpha \left( \frac{\|c\|_X + \|d\|_X}{b-a} \right) \right) \cdot \left| x - x' \right|
\end{aligned}$$

and

$$\begin{aligned}
\left\| F_m(x) - F_m\left(x'\right) \right\|_X &\leq \gamma \cdot \left| x - x' \right| + \alpha \cdot \left\| y_m(x) - y_m\left(x'\right) \right\|_X + \\
&+ \beta \cdot \left\| z_m(x) - z_m\left(x'\right) \right\|_X \leq (\gamma + \alpha\delta + \beta M) \cdot \left| x - x' \right| = L_0 \cdot \left| x - x' \right|
\end{aligned}$$

for any $x, x' \in [a,b]$ and $m \in \mathbb{N}$ with

$$L_0 = \gamma + \alpha \left( \frac{\|c\|_X + \|d\|_X}{b-a} + \frac{M\,(b-a)^2}{2} \right) + \beta M.$$

On the other hand,

$$\begin{aligned}
&\left\| H_{m,i}(s) - H_{m,i}(s') \right\|_X = \left\| G(x_i,s) \cdot f(s, y_m(s), z_m(s)) - G(x_i,s') \cdot f(s', y_m(s'), z_m(s')) \right\|_X \\
&\quad \leq M \cdot \left| G(x_i,s) - G(x_i,s') \right| + \left| G(x_i,s') \right| \cdot \left\| f(s, y_m(s), z_m(s)) - f(s', y_m(s'), z_m(s')) \right\|_X \\
&\quad \leq \left[ M + \frac{b-a}{4} \cdot L_0 \right] \cdot \left| s - s' \right| = L_1 \cdot \left| s - s' \right|
\end{aligned}$$

and

$$\begin{aligned}
&\left\| K_{m,i}(s) - K_{m,i}(s') \right\|_X = \left\| \frac{\partial G}{\partial x}(x_i,s) \cdot f(s, y_m(s), z_m(s)) - \frac{\partial G}{\partial x}(x_i,s') \cdot f(s', y_m(s'), z_m(s')) \right\|_X \\
&\leq M \cdot \left| \frac{\partial G}{\partial x}(x_i,s) - \frac{\partial G}{\partial x}(x_i,s') \right| + \left| \frac{\partial G}{\partial x}(x_i,s') \right| \cdot \left\| f(s, y_m(s), z_m(s)) - f(s', y_m(s'), z_m(s')) \right\|_X \\
&\leq \left( \frac{M}{b-a} + L_0 \right) \cdot \left| s - s' \right| = L_2 \cdot \left| s - s' \right|
\end{aligned}$$

for any $s, s' \in [a,b]$ and $m \in \mathbb{N}$ with

$$L_1 = M + \frac{(b-a)}{4} \cdot \left[ \gamma + \alpha \left( \frac{\|c\|_X + \|d\|_X}{b-a} + \frac{M\,(b-a)^2}{2} \right) + \beta M \right],$$

$$L_2 = \frac{M}{b-a} + \gamma + \alpha \left( \frac{\|c\|_X + \|d\|_X}{b-a} + \frac{M\,(b-a)^2}{2} \right) + \beta M.$$

□

# 4 The algorithm

To compute the Bochner integrals from (3.10) and (3.11) we apply the trapezoidal quadrature rule on Banach spaces (see [5] and [7]):

$$\int_a^b F(x)\,\mathrm{d}x = \frac{(b-a)}{2n}\cdot\sum_{i=1}^{n}\left[F\left(a+\frac{i\cdot(b-a)}{n}\right)+F\left(a+\frac{(i-1)\cdot(b-a)}{n}\right)\right]+R_n(F) \quad (4.1)$$

with the remainder estimation

$$\|R_n(F)\|_X \le \begin{cases} \frac{L(b-a)^2}{4n} & \text{if } F\in Lip\left([a,b],X\right), \ \text{(see [5])} \\ \frac{(b-a)^2}{4n}\cdot\|F'\|_C & \text{if } F\in C^1\left([a,b],X\right), \ \text{(see [7]).}\end{cases} \quad (4.2)$$

The estimation in the case $X=\mathbb{R}$ was obtained in [9].

Applying the quadrature rule (4.1)-(4.2) to the integrals from (3.10) and (3.11) on the partition (3.9) we obtain the following numerical method:

$$y_0(x_i)=\frac{x_i-a}{b-a}\cdot d+\frac{b-x_i}{b-a}\cdot c, \qquad z_0(x_i)=\frac{d-c}{b-a}, \quad i=\overline{0,n}$$

$$y_m(x_0)=c, \qquad y_m(x_n)=d, \qquad m\in\mathbb{N}^*$$

$$\begin{aligned} y_m(x_i) &= \frac{x_i-a}{b-a}\cdot d+\frac{b-x_i}{b-a}\cdot c-\int_a^b H_{m-1,i}(s)\,\mathrm{d}s \\ &= \frac{x_i-a}{b-a}\cdot d+\frac{b-x_i}{b-a}\cdot c-\frac{b-a}{2n}\cdot\sum_{j=1}^{n}\left[H_{m-1,i}(x_j)+H_{m-1,i}(x_{j-1})\right]+R_{m,i}\,, \\ & \qquad\qquad i=\overline{1,n-1},\ m\in\mathbb{N}^*, \\ z_m(x_i) &= \frac{d-c}{b-a}-\int_a^b K_{m-1,i}(s)\,\mathrm{d}s \\ &= \frac{d-c}{b-a}-\frac{b-a}{2n}\cdot\sum_{j=1}^{n}\Big[K_{m-1,i}(x_j)+K_{m-1,i}(x_{j-1})\Big]+\omega_{m,i}\,,\ i=\overline{0,n},\ m\in\mathbb{N}^*, \end{aligned}$$

with

$$\|R_{m,i}\|_X \le \frac{(b-a)^2\cdot L_1}{4n}, \qquad \|\omega_{m,i}\|_X\le\frac{(b-a)^2\cdot L_2}{4n}, \quad i=\overline{0,n},\ m\in\mathbb{N}^*. \quad (4.3)$$

These lead to the following algorithm:

$$y_0(x_i)=\frac{x_i-a}{b-a}\cdot d+\frac{b-x_i}{b-a}\cdot c, \qquad z_0(x_i)=\frac{d-c}{b-a}, \quad i=\overline{0,n} \quad (4.4)$$

$$y_m(x_0)=c, \qquad y_m(x_n)=d, \quad m\in\mathbb{N}^* \quad (4.5)$$

$$\begin{aligned} y_1(x_i) &= \frac{x_i-a}{b-a}\cdot d+\frac{b-x_i}{b-a}\cdot c-\frac{(b-a)}{2n}\cdot\sum_{j=1}^{n}\big[G(x_i,x_j)\cdot f(x_j,y_0(x_j),z_0(x_j)) \\ &+ G(x_i,x_{j-1})\cdot f(x_{j-1},y_0(x_{j-1}),z_0(x_{j-1}))\big]+R_{1,i}=\overline{y_1(x_i)}+R_{1,i}, \ \ i=\overline{1,n-1} \quad (4.6)\end{aligned}$$

$$z_1(x_i) = \frac{d-c}{b-a} - \frac{(b-a)}{2n} \cdot \sum_{j=1}^{n} \left[ \frac{\partial G}{\partial x}(x_i, x_j) \cdot f(x_j, y_0(x_j), z_0(x_j)) + \right.$$

$$\left. + \frac{\partial G}{\partial x}(x_i, x_{j-1}) \cdot f(x_{j-1}, y_0(x_{j-1}), z_0(x_{j-1})) \right] + \omega_{1,i} = \overline{z_1(x_i)} + \omega_{1,i}, \quad i = \overline{0,n} \tag{4.7}$$

$$\begin{aligned} y_2(x_i) = & \frac{x_i - a}{b-a} \cdot d + \frac{b - x_i}{b-a} \cdot c - \frac{b-a}{2n} \cdot \sum_{j=1}^{n} \Big[ G(x_i, x_j) \cdot f(x_j, \overline{y_1(x_j)} + R_{1,j}, \overline{z_1(x_j)} + \omega_{1,j}) \\ & + G(x_i, x_{j-1}) \cdot f\left( x_{j-1}, \overline{y_1(x_{j-1})} + R_{1,j-1}, \overline{z_1(x_{j-1})} + \omega_{1,j-1} \right) \Big] + R_{2,i} \\ = & \frac{x_i - a}{b-a} \cdot d + \frac{b - x_i}{b-a} \cdot c - \sum_{j=1}^{n} \Big[ G(x_i, x_j) \cdot f\left( x_j, \overline{y_1(x_j)}, \overline{z_1(x_j)} \right) \\ & + G(x_i, x_{j-1}) \cdot f\left( x_{j-1}, \overline{y_1(x_{j-1})}, \overline{z_1(x_{j-1})} \right) \Big] + \overline{R_{2,i}} \\ = & \overline{y_2(x_i)} + \overline{R_{2,i}}, \quad i = \overline{1, n-1} \end{aligned} \tag{4.8}$$

$$\begin{aligned} z_2(x_i) = & \frac{d-c}{b-a} - \frac{b-a}{2n} \cdot \sum_{j=1}^{n} \left[ \frac{\partial G}{\partial x}(x_i, x_j) \cdot f\left( x_j, \overline{y_1(x_j)} + R_{1,j}, \overline{z_1(x_j)} + \omega_{1,j} \right) + \right. \\ & \left. + \frac{\partial G}{\partial x}(x_i, x_{j-1}) \cdot f\left( x_{j-1}, \overline{y_1(x_{j-1})} + R_{1,j-1}, \overline{z_1(x_{j-1})} + \omega_{1,j-1} \right) \right] + \omega_{2,i} \\ = & \frac{d-c}{b-a} - \frac{(b-a)}{2n} \cdot \sum_{j=1}^{n} \left[ \frac{\partial G}{\partial x}(x_i, x_j) \cdot f\left( x_j, \overline{y_1(x_j)}, \overline{z_1(x_j)} \right) + \right. \\ & \left. + \frac{\partial G}{\partial x}(x_i, x_{j-1}) \cdot f\left( x_{j-1}, \overline{y_1(x_{j-1})}, \overline{z_1(x_{j-1})} \right) \right] + \overline{\omega_{2,i}} = \overline{z_2(x_i)} + \overline{\omega_{2,i}}, \quad i = \overline{0,n} \end{aligned} \tag{4.9}$$

and by induction for $m \geq 3$,

$$\begin{aligned} y_m(x_i) = & \frac{x_i - a}{b-a} \cdot d + \frac{b - x_i}{b-a} \cdot c \\ & - \frac{b-a}{2n} \cdot \sum_{j=1}^{n} \Big[ G(x_i, x_j) \cdot f(x_j, \overline{y_{m-1}(x_j)} + \overline{R_{m-1,j}}, \overline{z_{m-1}(x_j)} + \overline{\omega_{m-1,j}}) \\ & + G(x_i, x_{j-1}) \cdot f(x_{j-1}, \overline{y_{m-1}(x_{j-1})} + \overline{R_{m-1,j-1}}, \overline{z_{m-1}(x_{j-1})} + \overline{\omega_{m-1,j-1}}) \Big] + R_{m,i} \\ = & \frac{x_i - a}{b-a} \cdot d + \frac{b - x_i}{b-a} \cdot c - \frac{b-a}{2n} \cdot \sum_{j=1}^{n} \Big[ G(x_i, x_j) \cdot f\left( x_j, \overline{y_{m-1}(x_j)}, \overline{z_{m-1}(x_j)} \right) \\ & + G(x_i, x_{j-1}) \cdot f\left( x_{j-1}, \overline{y_{m-1}(x_{j-1})}, \overline{z_{m-1}(x_{j-1})} \right) \Big] + \overline{R_{m,i}} = \overline{y_m(x_i)} + \overline{R_{m,i}}, \quad i = \overline{1, n-1} \end{aligned} \tag{4.10}$$

$$\begin{aligned} z_m(x_i) = & \frac{d-c}{b-a} - \frac{b-a}{2n} \cdot \sum_{j=1}^{n} \left[ \frac{\partial G}{\partial x}(x_i, x_j) \cdot f\left( x_j, \overline{y_{m-1}(x_j)} + \overline{R_{m-1,j}}, \overline{z_{m-1}(x_j)} + \overline{\omega_{m-1,j}} \right) \right. \\ & \left. + \frac{\partial G}{\partial x}(x_i, x_{j-1}) \cdot f\left( x_{j-1}, \overline{y_{m-1}(x_{j-1})} + \overline{R_{m-1,j-1}}, \overline{z_{m-1}(x_{j-1})} + \overline{\omega_{m-1,j-1}} \right) \right] + \omega_{m,i} \end{aligned}$$

$$= \frac{d-c}{b-a} - \frac{(b-a)}{2n} \cdot \sum_{j=1}^{n} \left[ \frac{\partial G}{\partial x}(x_i, x_j) \cdot f\left(x_j, \overline{y_{m-1}(x_j)}, \overline{z_{m-1}(x_j)}\right)\right.$$

$$\left. + \frac{\partial G}{\partial x}(x_i, x_{j-1}) \cdot f\left(x_{j-1}, \overline{y_{m-1}(x_{j-1})}, \overline{z_{m-1}(x_{j-1})}\right)\right] + \overline{\omega_{m,i}} = \overline{z_m(x_i)} + \overline{\omega_{m,i}}, \quad i = \overline{0,n}. \tag{4.11}$$

The effective computed values are $\overline{y_m(x_i)}$, $\overline{z_m(x_i)}$, $i = \overline{0,n}$ approximating on the knots (3.9) the solution of the system (1.3).

# 5 The convergence analysis

## 5.1 The error estimation

**Theorem 5.1** *Under the conditions (i)-(iv), if* $\frac{\alpha}{4}(b-a)^2 + \beta(b-a) < 1$, *then the effective computed values* $\overline{y_m(x_i)}$, $\overline{z_m(x_i)}$, $i = \overline{0,n}$, $m \in \mathbb{N}^*$ *approximate the solution of the system (1.3) on the knots (3.9) with the apriori error estimate:*

$$\begin{pmatrix} \left\| y^*(t_i) - \overline{y_m(x_i)} \right\|_X \\ \left\| (y^*)'(t_i) - \overline{z_m(x_i)} \right\|_X \end{pmatrix} \le \frac{\left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]^{m-1}}{1 - \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]} \cdot \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} (b-a)^2 \cdot \frac{M_0}{8} \\ (b-a) \cdot \frac{M_0}{2} \end{pmatrix} + \begin{pmatrix} 1 - \frac{\alpha(b-a)^2}{4} & -\frac{\beta(b-a)^2}{4} \\ -\alpha(b-a) & 1 - \beta(b-a) \end{pmatrix}^{-1} \cdot \begin{pmatrix} \frac{(b-a)^2 \cdot L_1}{4n} \\ \frac{(b-a)^2 \cdot L_2}{4n} \end{pmatrix} \tag{5.1}$$

*for any* $i = \overline{0,n}$ *and* $m \in \mathbb{N}^*$.

*Proof.* According to the inequality (3.3) we have

$$\begin{pmatrix} \left\| y^*(t_i) - \overline{y_m(x_i)} \right\|_X \\ \left\| (y^*)'(t_i) - \overline{z_m(x_i)} \right\|_X \end{pmatrix} \le \begin{pmatrix} \| y^*(t_i) - y_m(x_i) \|_X \\ \left\| (y^*)'(t_i) - z_m(x_i) \right\|_X \end{pmatrix} + \begin{pmatrix} \left\| y_m(x_i) - \overline{y_m(x_i)} \right\|_X \\ \left\| z_m(x_i) - \overline{z_m(x_i)} \right\|_X \end{pmatrix} \le$$

$$\le \frac{\left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]^{m-1}}{1 - \left[\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a)\right]} \cdot Q \cdot \begin{pmatrix} (b-a)^2 \cdot \frac{M_0}{8} \\ (b-a) \cdot \frac{M_0}{2} \end{pmatrix} + \begin{pmatrix} \left\| \overline{R_{m,i}} \right\|_X \\ \left\| \overline{\omega_{m,i}} \right\|_X \end{pmatrix}, \quad i = \overline{0,n}, \ m \in \mathbb{N}^*.$$

In order to estimate the remainders $\overline{R_{m,i}}$, $\overline{\omega_{m,i}}$, $i = \overline{0,n}$, $m \in \mathbb{N}^*$ we obtain from (4.10), (4.11) the recurrences:

$$\left\| \overline{R_{m,i}} \right\|_X \le \| R_{m,i} \|_X + \frac{(b-a)}{2n} \cdot \sum_{j=1}^{n} [G(x_i, x_j) \cdot \left(\alpha \cdot \left\| \overline{R_{m-1,j}} \right\|_X + \beta \cdot \| \overline{\omega_{m-1,j}} \|_X\right) +$$

$$+ G(x_i, x_{j-1}) \cdot \left(\alpha \cdot \left\| \overline{R_{m-1,j-1}} \right\|_X + \beta \cdot \| \overline{\omega_{m-1,j-1}} \|_X\right)], \quad i = \overline{0,n}, \ m \in \mathbb{N}^*,$$

$$\| \overline{\omega_{m,i}} \|_X \le \| \omega_{m,i} \|_X + \frac{(b-a)}{2n} \cdot \sum_{j=1}^{n} [\frac{\partial G}{\partial x}(x_i, x_j) \cdot \left(\alpha \cdot \left\| \overline{R_{m-1,j}} \right\|_X + \beta \cdot \| \overline{\omega_{m-1,j}} \|_X\right) +$$

$$+ \frac{\partial G}{\partial x}(x_i, x_{j-1}) \cdot \left(\alpha \cdot \left\| \overline{R_{m-1,j-1}} \right\|_X + \beta \cdot \| \overline{\omega_{m-1,j-1}} \|_X\right)], \quad i = \overline{0,n}, \ m \in \mathbb{N}^*$$

and since

$$|G(x,s)| \le \frac{(b-a)}{4}, \qquad \left|\frac{\partial G}{\partial x}(x,s)\right| \le 1, \quad \forall x,s \in [a,b]$$

using the estimates (4.3), it follows for any $m \in \mathbb{N}^*$ that:

$$\begin{pmatrix} \left\|\overline{R_{m,i}}\right\|_X \\ \left\|\overline{\omega_{m,i}}\right\|_X \end{pmatrix} \le \begin{pmatrix} \frac{(b-a)^2 \cdot L_1}{4n} \\ \frac{(b-a)^2 \cdot L_2}{4n} \end{pmatrix} + \begin{pmatrix} \frac{\alpha(b-a)^2}{4} & \frac{\beta(b-a)^2}{4} \\ \alpha(b-a) & \beta(b-a) \end{pmatrix} \cdot \begin{pmatrix} \left\|\overline{R_{m-1}}\right\|_X \\ \left\|\overline{\omega_{m-1}}\right\|_X \end{pmatrix}, \quad i=\overline{0,n}$$

where

$$\left\|\overline{R_{m-1}}\right\|_X = \max\left\{\left\|\overline{R_{m-1,j}}\right\|_X : j=\overline{0,n}\right\}, \qquad \left\|\overline{\omega_{m-1}}\right\|_X = \max\left\{\left\|\overline{\omega_{m-1}}\right\|_X : j=\overline{0,n}\right\}.$$

Then,

$$\begin{pmatrix} \left\|\overline{R_{2,i}}\right\|_X \\ \left\|\overline{\omega_{2,i}}\right\|_X \end{pmatrix} \le \begin{pmatrix} \frac{(b-a)^2 \cdot L_1}{4n} \\ \frac{(b-a)^2 \cdot L_2}{4n} \end{pmatrix} + \begin{pmatrix} \frac{\alpha(b-a)^2}{4} & \frac{\beta(b-a)^2}{4} \\ \alpha(b-a) & \beta(b-a) \end{pmatrix} \cdot \begin{pmatrix} \frac{(b-a)^2 \cdot L_1}{4n} \\ \frac{(b-a)^2 \cdot L_2}{4n} \end{pmatrix} =$$
$$= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + Q_2\right] \cdot \begin{pmatrix} \frac{(b-a)^2 \cdot L_1}{4n} \\ \frac{(b-a)^2 \cdot L_2}{4n} \end{pmatrix}, \quad i=\overline{0,n}$$

where we wrote $Q_2 = \begin{pmatrix} \frac{\alpha(b-a)^2}{4} & \frac{\beta(b-a)^2}{4} \\ \alpha(b-a) & \beta(b-a) \end{pmatrix}$, and

$$\begin{pmatrix} \left\|\overline{R_{3,i}}\right\|_X \\ \left\|\overline{\omega_{3,i}}\right\|_X \end{pmatrix} \le \begin{pmatrix} \frac{(b-a)^2 \cdot L_1}{4n} \\ \frac{(b-a)^2 \cdot L_2}{4n} \end{pmatrix} + Q_2 \cdot \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + Q_2\right] \cdot \begin{pmatrix} \frac{(b-a)^2 \cdot L_1}{4n} \\ \frac{(b-a)^2 \cdot L_2}{4n} \end{pmatrix} =$$
$$= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + Q_2 + Q_2^2\right] \cdot \begin{pmatrix} \frac{(b-a)^2 \cdot L_1}{4n} \\ \frac{(b-a)^2 \cdot L_2}{4n} \end{pmatrix}, \quad i=\overline{0,n}.$$

So, by induction for $m \ge 3$ we obtain,

$$\begin{pmatrix} \left\|\overline{R_{m,i}}\right\|_X \\ \left\|\overline{\omega_{m,i}}\right\|_X \end{pmatrix} \le \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + Q_2 + \cdots + Q_2^{m-1}\right] \cdot \begin{pmatrix} \frac{(b-a)^2 \cdot L_1}{4n} \\ \frac{(b-a)^2 \cdot L_2}{4n} \end{pmatrix}$$
$$\le \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + Q_2 + \cdots + Q_2^{m-1} + \dots\right] \cdot \begin{pmatrix} \frac{(b-a)^2 \cdot L_1}{4n} \\ \frac{(b-a)^2 \cdot L_2}{4n} \end{pmatrix}$$
$$= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - Q_2\right]^{-1} \cdot \begin{pmatrix} \frac{(b-a)^2 \cdot L_1}{4n} \\ \frac{(b-a)^2 \cdot L_2}{4n} \end{pmatrix}, \quad i=\overline{0,n}$$

because the eigenvalues of the matrix $Q_2 = \begin{pmatrix} \frac{\alpha(b-a)^2}{4} & \frac{\beta(b-a)^2}{4} \\ \alpha(b-a) & \beta(b-a) \end{pmatrix}$ are $\lambda_1 = 0$ and $\lambda_2 = \frac{\alpha}{4}(b-a)^2 + \beta(b-a) < 1$ so that

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + Q_2 + \cdots + Q_2^{m-1} + \dots\right] = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{\alpha(b-a)^2}{4} & \frac{\beta(b-a)^2}{4} \\ \alpha(b-a) & \beta(b-a) \end{pmatrix}\right]^{-1}.$$

□

**Remark 5.2** *From the estimates (3.3) and (5.1) we infer that*

$$\lim_{m\to\infty,n\to\infty}\left(\begin{array}{c}\left\|y^*(t_i)-\overline{y_m(x_i)}\right\|_X\\ \left\|(y^*)'(t_i)-\overline{z_m(x_i)}\right\|_X\end{array}\right)=\left(\begin{array}{c}0\\0\end{array}\right)$$

*that is the convergence of the algorithm to the solution of the system (1.3).*

**Remark 5.3** *We see that the apriori (3.3) and a posteriori (3.4) error estimates can offer a practical stopping criterion of the algorithm. This can be stated as follows: for given* $\varepsilon'>0$ *and given* $n\in\mathbb{N}^*$ *(previously chosen) we determine the first natural number* $m\in\mathbb{N}^*$ *for which*

$$\left\|\overline{y_m(x_i)}-\overline{y_{m-1}(x_i)}\right\|_X<\varepsilon',\ \textit{for all } i=\overline{1,n-1}$$

*and*

$$\left\|\overline{z_m(x_i)}-\overline{z_{m-1}(x_i)}\right\|_X<\varepsilon',\ \textit{for all } i=\overline{0,n}$$

*and we stop to this* $m$ *retaining the approximations* $\overline{y_m(x_i)}$, $\overline{z_m(x_i)}$, $i=\overline{0,n}$ *of the solution. The demonstration of this criterion is the following. We denote*

$$\Omega=\left(\begin{array}{cc}1-\frac{\alpha(b-a)^2}{4} & -\frac{\beta(b-a)^2}{4}\\ -\alpha(b-a) & 1-\beta(b-a)\end{array}\right)^{-1}\cdot\left(\begin{array}{c}\frac{(b-a)^2\cdot L_1}{4n}\\ \frac{(b-a)^2\cdot L_2}{4n}\end{array}\right)$$

*and we have*

$$\left(\begin{array}{c}\left\|y^*(t_i)-\overline{y_m(x_i)}\right\|_X\\ \left\|(y^*)'(t_i)-\overline{z_m(x_i)}\right\|_X\end{array}\right)\le\left(\begin{array}{c}\|y^*(t_i)-y_m(x_i)\|_X\\ \|(y^*)'(t_i)-z_m(x_i)\|_X\end{array}\right)+\left(\begin{array}{c}\left\|y_m(x_i)-\overline{y_m(x_i)}\right\|_X\\ \left\|z_m(x_i)-\overline{z_m(x_i)}\right\|_X\end{array}\right)$$

$$\le\frac{\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)}{1-\left[\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)\right]}\cdot Q\cdot\left(\begin{array}{c}\|y_m(x_i)-y_{m-1}(x_i)\|_X\\ \|z_m(x_i)-z_{m-1}(x_i)\|_X\end{array}\right)+\left(\begin{array}{c}\left\|\overline{R_{m,i}}\right\|_X\\ \left\|\overline{\omega_{m,i}}\right\|_X\end{array}\right)$$

*and*

$$\begin{aligned}\begin{pmatrix}\|y_m(x_i)-y_{m-1}(x_i)\|_X\\ \|z_m(x_i)-z_{m-1}(x_i)\|_X\end{pmatrix}&\le\begin{pmatrix}\left\|y_m(x_i)-\overline{y_m(x_i)}\right\|_X\\ \left\|z_m(x_i)-\overline{z_m(x_i)}\right\|_X\end{pmatrix}+\\ &+\begin{pmatrix}\left\|\overline{y_m(x_i)}-\overline{y_{m-1}(x_i)}\right\|_X\\ \left\|\overline{z_m(x_i)}-\overline{z_{m-1}(x_i)}\right\|_X\end{pmatrix}+\begin{pmatrix}\left\|\overline{y_{m-1}(x_i)}-y_{m-1}(x_i)\right\|_X\\ \left\|\overline{z_{m-1}(x_i)}-z_{m-1}(x_i)\right\|_X\end{pmatrix}\\ &\le\begin{pmatrix}\left\|\overline{R_{m,i}}\right\|_X\\ \left\|\overline{\omega_{m,i}}\right\|_X\end{pmatrix}+\begin{pmatrix}\left\|\overline{R_{m-1,i}}\right\|_X\\ \left\|\overline{\omega_{m-1,i}}\right\|_X\end{pmatrix}+\begin{pmatrix}\left\|\overline{y_m(x_i)}-\overline{y_{m-1}(x_i)}\right\|_X\\ \left\|\overline{z_m(x_i)}-\overline{z_{m-1}(x_i)}\right\|_X\end{pmatrix}.\end{aligned}$$

*So,*

$$\left(\begin{array}{c}\left\|y^*(t_i)-\overline{y_m(x_i)}\right\|_X\\ \left\|(y^*)'(t_i)-\overline{z_m(x_i)}\right\|_X\end{array}\right)\le\left(\begin{array}{c}\left\|\overline{R_{m,i}}\right\|_X\\ \left\|\overline{\omega_{m,i}}\right\|_X\end{array}\right)+$$

$$+\frac{\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)}{1-\left[\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)\right]}\cdot\left(\begin{array}{cc}\frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2\\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a)\end{array}\right)\cdot\left(\begin{array}{c}\left\|\overline{y_m(x_i)}-\overline{y_{m-1}(x_i)}\right\|_X\\ \left\|\overline{z_m(x_i)}-\overline{z_{m-1}(x_i)}\right\|_X\end{array}\right)$$

$$+\frac{\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)}{1-\left[\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)\right]}\cdot Q\cdot\left[\left(\begin{array}{c}\left\|\overline{R_{m,i}}\right\|_X\\ \left\|\overline{\omega_{m,i}}\right\|_X\end{array}\right)+\left(\begin{array}{c}\left\|\overline{R_{m-1,i}}\right\|_X\\ \left\|\overline{\omega_{m-1,i}}\right\|_X\end{array}\right)\right].$$

*Then*

$$\left(\begin{array}{c}\left\|y^*(t_i)-\overline{y_m(x_i)}\right\|_X\\ \left\|(y^*)'(t_i)-\overline{z_m(x_i)}\right\|_X\end{array}\right)\le\frac{1+\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)}{1-\left[\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)\right]}\cdot\left(\begin{array}{cc}\frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2\\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a)\end{array}\right)\cdot\Omega$$

$$+\frac{\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)}{1-\left[\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)\right]}\cdot\left(\begin{array}{cc}\frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2\\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a)\end{array}\right)\cdot\left(\begin{array}{c}\left\|\overline{y_m(x_i)}-\overline{y_{m-1}(x_i)}\right\|_X\\ \left\|\overline{z_m(x_i)}-\overline{z_{m-1}(x_i)}\right\|_X\end{array}\right).$$

*For given $\varepsilon>0$ we require*

$$\frac{1+\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)}{1-\left[\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)\right]}\cdot\left(\begin{array}{cc}\frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2\\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a)\end{array}\right)\cdot\Omega<\left(\begin{array}{c}\frac{\varepsilon}{2}\\ \frac{\varepsilon}{2}\end{array}\right)\tag{5.2}$$

*and*

$$\frac{\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)}{1-\left[\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)\right]}\cdot\left(\begin{array}{cc}\frac{\alpha(b-a)^2}{8} & \frac{\beta(b-a)^2}{8}\\ \frac{\alpha(b-a)}{2} & \frac{\beta(b-a)}{2}\end{array}\right)\cdot\left(\begin{array}{c}\left\|\overline{y_m(x_i)}-\overline{y_{m-1}(x_i)}\right\|_X\\ \left\|\overline{z_m(x_i)}-\overline{z_{m-1}(x_i)}\right\|_X\end{array}\right)<\left(\begin{array}{c}\frac{\varepsilon}{2}\\ \frac{\varepsilon}{2}\end{array}\right).\tag{5.3}$$

*From inequality (5.2) we determine the smallest natural number $n$ for which this inequality holds. Afterwards, we find the smallest natural number $m$ for which the inequality (5.3) holds.*

**Remark 5.4** *Comparing the hypotheses in Theorems 3.1 and 5.1 we see that in Theorem 5.1 only the supplementary Lipschitz condition (iv) appears and the inequality $\frac{\alpha}{8}(b-a)^2+\frac{\beta}{2}(b-a)<1$ becomes $\frac{\alpha}{4}(b-a)^2+\beta(b-a)<1$.*

## 5.2 The numerical stability

In order to obtain the numerical stability of the method we consider the two-point boundary value problem with the same second order differential equation, but with modified boundary values:

$$\left\{\begin{array}{ll}y''(x)=f\left(x,y(x),y'(x)\right), & x\in[a,b]\\ y(a)=c',\quad y(b)=d'.\end{array}\right.\tag{5.4}$$

such that $\|c-c'\|_X<\epsilon$ and $\|d-d'\|_X<\varepsilon$.

For the boundary value problem (5.4) the sequence of successive approximations on the same knots is:

$$v_0(x_i)=\frac{x_i-a}{b-a}\cdot d'+\frac{b-x_i}{b-a}\cdot c',\qquad w_0(x_i)=\frac{d'-c'}{b-a},\quad i=\overline{0,n}$$

$$v_m(x_0)=c',\qquad v_m(x_n)=d',\quad m\in\mathbb{N}^*$$

$$v_m(x_i)=\frac{x_i-a}{b-a}\cdot d+\frac{b-x_i}{b-a}\cdot c-\int_a^b G(x_i,s)\cdot f\left(s,v_{m-1}(s),w_{m-1}(s)\right)\,\mathrm{d}s,\quad i=\overline{1,n-1},\ m\in\mathbb{N}^*$$

$$w_m(x_i) = \frac{d-c}{b-a} - \int_a^b \frac{\partial G}{\partial x}(x_i, s) \cdot f(s, v_{m-1}(s), w_{m-1}(s))\, \mathrm{d}s, \quad i = \overline{0,n},\ m \in \mathbb{N}^*$$

and the effective computed values are

$$v_0(x_i) = \frac{x_i - a}{b-a} \cdot d + \frac{b - x_i}{b-a} \cdot c, \qquad w_0(x_i) = \frac{d-c}{b-a}, \quad i = \overline{0,n}$$

$$v_m(x_0) = c', \qquad v_m(x_n) = d', \quad m \in \mathbb{N}^*,$$

and $\overline{v_m(x_i)}$, $i = \overline{1, n-1}$, $\overline{w_m(x_i)}$, $i = \overline{0,n}$, $m \in \mathbb{N}^*$ with $v_m(x_i) = \overline{v_m(x_i)} + \overline{R'_{m,i}}$ and $w_m(x_i) = \overline{w_m(x_i)} + \overline{\omega'_{m,i}}$. We see that

$$\|y_0(x) - v_0(x)\|_X \le \|d - d'\|_X + \|c - c'\|_X < \epsilon + \varepsilon, \text{ for all } x \in [a,b]$$

and

$$\|z_0(x) - w_0(x)\|_X \le \frac{1}{b-a} \cdot \left(\|d - d'\|_X + \|c - c'\|_X\right) < \frac{\epsilon + \varepsilon}{b-a}, \text{ for all } x \in [a,b].$$

**Definition 5.5** *We say that the proposed method is numerically stable if there exist $p \in \mathbb{N}^*$ and the matrices $K_1, K_2, K_3 \in \mathbb{R}^2_+$ such that*

$$\begin{pmatrix} \left\|\overline{y_m(x_i)} - \overline{v_m(x_i)}\right\|_X \\ \left\|\overline{z_m(x_i)} - \overline{w_m(x_i)}\right\|_X \end{pmatrix} \le K_1 \cdot \epsilon + K_2 \cdot \varepsilon + K_3 \cdot h^p$$

*for all $i = \overline{0,n}$, $m \in \mathbb{N}^*$, where $h = \frac{b-a}{n}$.*

**Theorem 5.6** *Under the conditions of Theorem 5.1 the proposed method of successive approximations for the boundary value problem (1.1) is numerically stable.*

*Proof.* We have

$$\begin{pmatrix} \left\|\overline{y_m(x_i)} - \overline{v_m(x_i)}\right\|_X \\ \left\|\overline{z_m(x_i)} - \overline{w_m(x_i)}\right\|_X \end{pmatrix} \le \begin{pmatrix} \left\|\overline{y_m(x_i)} - y_m(x_i)\right\|_X \\ \left\|\overline{z_m(x_i)} - z_m(x_i)\right\|_X \end{pmatrix} + \begin{pmatrix} \|y_m(x_i) - v_m(x_i)\|_X \\ \|z_m(x_i) - w_m(x_i)\|_X \end{pmatrix} +$$

$$+ \begin{pmatrix} \left\|v_m(x_i) - \overline{v_m(x_i)}\right\|_X \\ \left\|w_m(x_i) - \overline{w_m(x_i)}\right\|_X \end{pmatrix} \le \begin{pmatrix} \|y_m(x_i) - v_m(x_i)\|_X \\ \|z_m(x_i) - w_m(x_i)\|_X \end{pmatrix} + \begin{pmatrix} \|\overline{R_{m,i}}\|_X \\ \|\overline{\omega_{m,i}}\|_X \end{pmatrix} + \begin{pmatrix} \left\|\overline{R'_{m,i}}\right\|_X \\ \left\|\overline{\omega'_{m,i}}\right\|_X \end{pmatrix}$$

and

$$\begin{pmatrix} \left\|\overline{R'_{m,i}}\right\|_X \\ \left\|\overline{\omega'_{m,i}}\right\|_X \end{pmatrix} \le \begin{pmatrix} 1 - \frac{\alpha(b-a)^2}{4} & -\frac{\beta(b-a)^2}{4} \\ -\alpha(b-a) & 1 - \beta(b-a) \end{pmatrix}^{-1} \cdot \begin{pmatrix} \frac{(b-a)^2 \cdot L_1}{4n} \\ \frac{(b-a)^2 \cdot L_2}{4n} \end{pmatrix}, \ \forall i = \overline{0,n}, m \in \mathbb{N}^*.$$

In inductive manner, according to the condition $\frac{\alpha}{8}(b-a)^2 + \frac{\beta}{2}(b-a) < 1$, we get:

$$\|y_0(x) - v_0(x)\|_X \le \|d - d'\|_X + \|c - c'\|_X < \epsilon + \varepsilon, \text{ for all } x \in [a,b],$$

$$\|z_0(x)-w_0(x)\|_X \le \frac{1}{b-a}\cdot\left(\|d-d'\|_X+\|c-c'\|_X\right) < \frac{\epsilon+\varepsilon}{b-a},\ \text{for all } x\in[a,b]$$

$$\|y_m(x_0)-v_m(x_0)\|_X \le \|c-c'\|_X < \epsilon$$

$$\|y_m(x_n)-v_m(x_n)\|_X \le \|d-d'\|_X < \varepsilon$$

and

$$\begin{pmatrix} \|y_m(x)-v_m(x)\|_X \\ \|z_m(x)-w_m(x)\|_X \end{pmatrix} \le \begin{pmatrix} \|y_0(x)-v_0(x)\|_X \\ \|z_0(x)-w_0(x)\|_X \end{pmatrix} +$$

$$+\begin{pmatrix} \int_a^b |G(x,s)|\cdot\|f(s,y_{m-1}(s),z_{m-1}(s))-f(s,v_{m-1}(s),w_{m-1}(s))\|_X\,\mathrm{d}s \\ \int_a^b \left|\frac{\partial G}{\partial x}(x,s)\right|\cdot\|f(s,y_{m-1}(s),z_{m-1}(s))-f(s,v_{m-1}(s),w_{m-1}(s))\|_X\,\mathrm{d}s \end{pmatrix} \le$$

$$\le \begin{pmatrix} \epsilon+\varepsilon \\ \frac{\epsilon+\varepsilon}{b-a} \end{pmatrix} + \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix}\cdot\begin{pmatrix} \|y_{m-1}(x)-v_{m-1}(x)\|_X \\ \|z_{m-1}(x)-w_{m-1}(x)\|_X \end{pmatrix} \le \dots \le$$

$$\le \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + Q + \dots + Q^m\right]\cdot\begin{pmatrix} \|y_0(x)-v_0(x)\|_X \\ \|z_0(x)-w_0(x)\|_X \end{pmatrix} \le$$

$$\le \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + Q + \dots + Q^m + \dots\right]\cdot\begin{pmatrix} \epsilon+\varepsilon \\ \frac{\epsilon+\varepsilon}{b-a} \end{pmatrix} =$$

$$= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix}\right]^{-1}\cdot\begin{pmatrix} \epsilon+\varepsilon \\ \frac{\epsilon+\varepsilon}{b-a} \end{pmatrix},\quad \forall x\in[a,b],\ \forall m\in\mathbb{N}^*,$$

where $Q = \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix}$. So,

$$\begin{pmatrix} \left\|\overline{y_m(x_i)}-\overline{v_m(x_i)}\right\|_X \\ \left\|\overline{z_m(x_i)}-\overline{w_m(x_i)}\right\|_X \end{pmatrix} \le \begin{pmatrix} \|y_m(x_i)-v_m(x_i)\|_X \\ \|z_m(x_i)-w_m(x_i)\|_X \end{pmatrix}$$

$$+\begin{pmatrix} 1-\frac{\alpha(b-a)^2}{4} & -\frac{\beta(b-a)^2}{4} \\ -\alpha(b-a) & 1-\beta(b-a) \end{pmatrix}^{-1}\cdot\begin{pmatrix} \frac{(b-a)^2\cdot L_1}{2n} \\ \frac{(b-a)^2\cdot L_2}{2n} \end{pmatrix}$$

$$\le \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{\alpha}{8}(b-a)^2 & \frac{\beta}{8}(b-a)^2 \\ \frac{\alpha}{2}(b-a) & \frac{\beta}{2}(b-a) \end{pmatrix}\right]^{-1}\cdot\begin{pmatrix} \epsilon+\varepsilon \\ \frac{\epsilon+\varepsilon}{b-a} \end{pmatrix}$$

$$+\frac{b-a}{2}\cdot\begin{pmatrix} 1-\frac{\alpha(b-a)^2}{4} & -\frac{\beta(b-a)^2}{4} \\ -\alpha(b-a) & 1-\beta(b-a) \end{pmatrix}^{-1}\cdot\frac{b-a}{n}\cdot\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$$

$$= K_1\cdot\epsilon + K_2\cdot\varepsilon + K_3\cdot h,\ \ \forall i=\overline{0,n},\ m\in\mathbb{N}^*.$$

Of course, in the same conditions the continuous dependence by data of the solution can be obtained by using an analogous technique. □

**Corollary 2** *The proposed method of successive approximations for the boundary value problem (1.1) is convergent.*

*Proof.* The convergence results from Theorem 5.1, Remark 5.2 and Theorem 5.6. □

## 6 Concluding remarks

For the two-point boundary value problem (1.1) considered in Banach spaces, the application of the Perov's fixed point theorem permits to obtain the existence, uniqueness and boundedness of the solution and to construct a convergent and stable approximation method for this solution. The convergence of the method can be proved using only Lipschitz conditions, without smoothness or boundedness conditions. These extend the applicability of the method. It is well-known that the existing methods such as shooting, collocation, projection and spline functions methods require at least the boundedness of the first order partial derivatives of the function $f$, usually imposing high order smoothness conditions. To illustrate the accuracy of the method we choose the case $X = \mathbb{R}$ and consider the following examples.

**Example 6.1** *The boundary value problem*

$$\begin{cases} y''(t) = 2y(x) + y'(x) - 3e^{-x} \\ y(0) = 0, \quad y(1) = e^{-1} \end{cases}, \quad x \in [0,1]$$

*has the exact solution $y^*(x) = xe^{-x}$ and applying the presented algorithm for $n = 10$ and for $n = 100$, we get the number of iterations $m = 19$. For $n = 1000$, the number of iterations is $m = 21$. The order of effective errors confirms the convergence of the method and it is presented in Table 1.*

**Example 6.2** *For the boundary value problem*

$$\begin{cases} y''(t) = -y(x) \cdot y'(x) + |y(x)|^3 \\ y(0) = 1, \quad y(1) = \frac{1}{2} \end{cases}, \quad x \in [0,1]$$

*the kernel function $f(s,u,v) = -u \cdot v + |u|^3$ is nonlinear and not differentiable. The exact solution is $y^*(x) = \frac{1}{x+1}$ and applying the presented algorithm, the error approximation results are in Table 1. For $n = 10$, $n = 100$, $n = 1000$, the number of iterations is $m = 33$, $m = 34$ and $m = 35$, respectively.*

**Example 6.3** *The boundary value problem*

$$\begin{cases} y''(t) = y(x) + y'(x) \\ y(0) = 1, \quad y(0.5) = \sqrt{e} \end{cases}, \quad x \in [0,0.5]$$

*has the exact solution $y^*(x) = e^x$ and the error approximation results are in Table 1. For $n = 10, 100$ and $1000$, we get the same number of iterations $m = 8$.*

In Table 1, in the second, third and fourth column we present the order of effective errors

$$er = \max\{ \left\| y^*(t_i) - \overline{y_m(x_i)} \right\|_X : i = \overline{1, n-1} \}$$

for the above presented three examples, corresponding to different stepsize $h = \frac{b-a}{n}$.

| $h$ | $er$, example 6.1 | $er$, example 6.2 | $er$, example 6.3 |
|---|---|---|---|
| 0.1 | $1.284 \times 10^{-4}$ | $2.51 \times 10^{-4}$ | $8.276 \times 10^{-5}$ |
| 0.01 | $1.254 \times 10^{-6}$ | $3.057 \times 10^{-6}$ | $8.648 \times 10^{-7}$ |
| 0.001 | $1.28 \times 10^{-8}$ | $3.064 \times 10^{-8}$ | $8.681 \times 10^{-9}$ |

Table 1

We see that for stepsize $h = 0.1$ the order of effective error is $O\left(10^{-4} \div 10^{-5}\right)$, for stepsize $h = 0.01$ this order is $O\left(10^{-6} \div 10^{-7}\right)$ and for stepsize $h = 0.001$ this order is $O\left(10^{-8} \div 10^{-9}\right)$. These confirm the convergence of the method. This method presented in Sections 3, 4 and 5 can be particularized for the cases $X = \mathbb{R}$ and $X = \mathbb{R}^n$, obtaining the method of successive approximations and the corresponding algorithm for scalar differential equations and for systems of differential equations, respectively.

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